The Algebraic Proof of the Universality Theorem

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1 Preliminary

In [LL1] the authors had developed a topological version of family Seiberg-Witten invariants. In an attempt to generalize Taubes' "SW=Gr" to families, it was observed that multiple coverings of exceptional curves may show up in the enumeration. In [Liu4] a family curve counting scheme has been proposed for algebraic families. In [L3] the topological/algebraic version of family blowup formula was derived. In [L5] the topological/algebraic version of family switching formula was derived. It was used in [Liu1] to derive the Göttsche-Yau-Zaslow conjecture regarding counting of nodal curves on algebraic surfaces, including $K3$.

The derivation in [Liu1] makes use of the various techniques from differential topology and complex geometry as well as the technique from the theory of pseudo-holomorphic curves. In the derivation the author had assumed that the reader is reasonably familiar with Taubes proof of "SW=Gr". On the other hand, a version of algebraic family Seiberg-Witten invariant was defined in [Liu3] and its family blowup formula and switching formula was derived in [Liu3] and [Liu5], respectively, based on intersection theory developed in [F].

Thus it is desirable to give a purely algebraic proof of the universality theorem, the backbone in deriving the Göttsche-Yau-Zaslow formula, based on intersection theory [F] and algebraic geometry [Ha].

The following universality theorem is the main theorem proved in this paper.

Main Theorem 1 *Let $\delta \in \mathbf{N}$ denote the number of nodal singularities. Let L be a $5\delta - 1$ very-ample line bundle on an algebraic surface M, then the number of $\delta-$node nodal singular curves in a generic δ dimensional linear sub-system of $|L|$ can be expressed as a universal polynomial (independent to M) of $c_1(L)^2$, $c_1(L) \cdot c_1(M)$, $c_1(M)^2$, $c_2(M)$ of degree δ.*

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For simplicity, we take $\mathbf{C}$ to be our ground field. But the same argument works for algebraic closed fields of characteristic zero as well.

The term "number of $\delta-$node nodal curves in a linear system" used here is the weighted number of singular algebraic curves with isolated $\delta-$nodal singularities defined in [Got]. This concept is closed related but not always equal to Gromov-Witten invariant of $c_1(L)$. For their potential difference, please consult section 7.

In the following we outline the different functionalities of the various sections of the paper. Because the paper is a bit lengthy–involving quite a few new notations not widely used in the community of enumeration geometry/algebraic geometry, in subsection 1.1 we offer some simple advice, a notation table and quite a lot of footnotes, which may help the reader to get used to our notations and read the paper more fluently.

In section 2, we review the basic facts about the universal spaces and then introduce the relative-universal spaces over a base B. We also recall the concept of admissible graphs, admissible strata and the stratification of the (relative) universal spaces. To minimize the dependence on [Liu1], we prove a few useful results which will be used in the latter sections. As a result, an admissible stratum can be characterized as the locus of co-existence of type I exceptional classes attached to it.

In section 3 and the subsidiary subsections, we develop a technique to construct and identify the quotient bundle $\mathbf{V}_{quot}$ of the given obstruction vector bundle $\mathbf{W}_{canon}|_{Y(\Gamma)\times T(M)}$, given some datum of quotient sheaves.

In section 4, we review the residual intersection formula of top Chern classes [F] and develop an algebraic tool to compare the top Chern classes of vector bundles $\sigma : E \mapsto F$ isomorphic off a closed subset. Under the bundle homomorphism σ, a section s_0 of E induces a section $\sigma(s_0)$ of F and the difference of their top Chern classes can be studied by the difference of their localized top Chern classes along the zero loci $Z(s_0),\ Z(\sigma(s_0))$, respectively. We find that after some blowing ups along loci in $\overline{Z(\sigma(s_0)) - Z(s_0)}$, the top Chern class of the residual bundle of F has the same "numerical property" as the top Chern class of the pull-back of E. We achieve this goal by a graph construction in the projective space bundle $\mathbf{P}(E \oplus 1)$. This proposition is crucial in controlling the seemingly "unmanageable" blowing ups and relate the modified bundle to better understood objects.

The canonical cross section s_{canon} of the canonical algebraic obstruction bundle [1] $\mathbf{H} \otimes \pi_X^* \mathbf{W}_{canon}$ defines a zero locus in $X = \mathbf{P}(\mathbf{V}_{canon})$ which contains the sub-locus (closure) $\overline{\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y_{\gamma_n}}$ that we want to study. The proof of the main theorem enables us to attach an invariant to this locus. In section 5 and the sub-sequential sub-sections, we initiate the proof of the main theorem by introducing an inductive procedure of blowing up the smooth total space of the canonical family algebraic Kuranishi space $X = \mathbf{P}(\mathbf{V}_{canon})$. The goal is to apply residual intersection theory inductively and remove all the excess contributions. In subsection 5.2, we define the modified algebraic family

[1] See definition 5.3 in [Liu3] for its definition.

Seiberg-Witten invariants attached to the various smooth sub-loci. We also address in subsection 5.3 some combinatorial questions regarding the sub-loci of $\mathcal{M}_{C-\mathbf{M}(E)E}$. In the same sub-section, we also explain the geometric meanings of several partial orderings $\gg, \succ, \sqsupset$, etc. which had already appeared in [Liu1], [Liu4], [Liu5] before.

The section 6 is the core of the current paper. In this section, we finish up the proof of the main theorem. We address in subsection 6.1 some combinatorial questions regarding the independence of the localized top Chern classes to the permutations/collapsing of the blowing up orderings. We prove inductively that the various excess contributions of the algebraic family invariant $\mathcal{AFSW}_{M_{\delta+1}\times\{t_L\}\mapsto M_\delta\times\{t_L\}}(1, C-2\sum_{1\leq i\leq\delta}E_i)$ can be identified with the various modified family algebraic Seiberg-Witten invariants defined in subsection 5.2. In subsection 6.4, we enhance Göttsche's argument slightly to get the necessary finiteness result on nodal curves that we enumerate.

Finally in the appendix, section 7, we offer a light-weighted comparison between our family invariants and the standard Gromov-Witten invariants of algebraic surfaces. Even though family Seiberg-Witten invariants on the universal spaces are related to Gromov-Witten invariants of algebraic surfaces, there is some subtle difference between them which may cause confusion to the readers who are new to family Seiberg-Witten invariants.

In this paper, we do not attempt to address the issue of identifying these universal polynomials in the universality theorem algebraically. The discussion about the relationship of the above universality theorem with Riemann-Roch formula along with some open problems and conjectures related to this theorem will be addressed in a separated note [Liu7] elsewhere.

Finally, it is recommended to use [Liu3], [Liu4], [Liu5] as companions reading this paper.

1.1 A Table of Our Notations

In this paper we give an algebraic proof of the universality theorem, and most of our notations have already appeared in [Liu1], [Liu3], [Liu4], [Liu5], etc. On the other hand a few new notations must be introduced in the purely algebraic argument of the paper. For the standard notations in algebraic geometry and intersection theory, the reader may consult [Ha], [F]. In the following, we list the frequently used "global" [2] notations in the paper. Following the convention of the earlier papers [Liu3], [Liu5], a locally free sheaf of sections will be denoted by the calligraphic character, say $\mathcal{G}$, if and only if the corresponding algebraic vector bundle has been denoted by the bold character $\mathbf{G}$.

Another convention in this paper is that the variables/notations defined within the proof of a lemma or a proposition is viewed as a "local" variable

[2] For its definition, see the next paragraph.

and its "scope" is within this particular proof. We may recycle the same variable/notation in the proofs of the other lemma/proposition in a different context. On the other hand, the variables/notations declared in the definitions are viewed as the "global" variables. Subject to some specialization of values, their meanings are fixed throughout the paper. Finally, the variables/notations declared in the text of the paper are "semi-local" in the sense that their scope is the whole section containing the particular text. If we refer to this variable/notation from a different section, we will indicate to the reader the location (page) where it has been defined.

The following is the list of notations widely used throughout the paper. Every symbol is leaded by a $\bullet$.

• $adm(n)$–the set of n-vertexes admissible graphs satisfying the five axioms starting at page 7.

• $adm_2(n)$–the subset of $adm(n)$ consisting of fan-like admissible graphs. See definition 6 on page 47 and fig.4 on page 47.

• C–the class in $H^{1,1}(M, \mathbf{Z})$. In this paper C is assumed to satisfy $\mathcal{R}^1\pi_*\big(\mathcal{E}_C\big) = \mathcal{R}^2\pi_*\big(\mathcal{E}_C\big) = 0$.

• $C - \mathbf{M}(E)E$–In this expression the term $-\mathbf{M}(E)E = -\sum_{1\leq i\leq n} m_iE_i$ is interpreted as a cohomology class instead of an effective divisor. $C - \mathbf{M}(E)E$ is a class of Hodge type $(1,1)$.

• $\mathcal{C}_\Gamma$–the simplicial exceptional cone constant over $\mathcal{S}_\Gamma \supset Y_\Gamma$.

• $\Delta(n)$–the subset of $adm(n)$ containing admissible graphs which satisfy the additional extremal conditions on page 47. See definition on page 47

• Δ_{k_i}–the (anti)-effective divisor in $\tilde{\Xi}_{k_i} \mapsto Y(\Gamma_{e_{k_i}})$ which appears in the exact sequence relating $\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}} \otimes \mathcal{E}_{-\mathbf{M}(E)E}\big)$ and $\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}}(e_{k_i})\big)$. Consult page 54 for more details.

• δ–the number of nodes in the main theorem.

• e_i–the extremal generators of $\mathcal{C}_\Gamma$, called type I exceptional class.

• E_i–the effective exceptional divisor $E_{i;n+1}$ in M_{n+1} and is viewed as the fiberwise divisor of $M_{n+1} \mapsto M_n$.

• $E_{a;b}$–the effective exceptional divisor in M_n corresponding to the blowing up along the $(a,b)-$th partial diagonal of M^n.

• $\mathcal{E}_C$–the invertible sheaf over $M \times T(M)$ with $c_1(\mathcal{E}_C|_{M\times\{t\}}) = C$ for $\forall t \in T(M)$.

• $\mathcal{E}_{C-\mathbf{M}(E)E}$–the invertible sheaf over $M_{n+1} \times T(M)$ corresponding to the cohomology class $C - \mathbf{M}(E)E$ of Hodge type $(1,1)$.

• $\mathcal{EC}_b(\underline{C}, Q)$–the type I exceptional cone associated with $\underline{C}$ over $b \in M_n$. See definition 5 of [Liu4] for its definition.

• f_n–the projection map $M_{n+1} \mapsto M_n$.

• $f_{n-1;k}$–the composition of the projection map $f_k \circ f_{k+1} \circ \cdots \circ f_{n-1} : M_n \mapsto M_k$.

• Γ–A typical element of an admissible graph $\Gamma \in adm(n)$. See fig. 1 on page 8 for an example.

• Γ_{e_i}–the fan-like admissible graph attached to e_i in which the index i is the only direct ascendent index and the indexes j_i appearing in $e_i = E_i - \sum_{j_i} E_{j_i}$ are the direct descendent indexes of i. See fig.2 on page 15 for some examples.

• γ_n or γ–the unique admissible graph of n vertexes with no one edge.

• $\mathbf{H}$ and $\mathcal{H}$–the hyperplane bundle and its invertible sheaf of sections on $\mathbf{P}(\mathcal{V}_{canon}) = \mathbf{P}(\mathbf{V}^{\circ}_{canon})$ induced by the linear structure $\mathbf{V}_{canon}$.

• I_Γ–the subset of $\Delta(n)$ which collects all the elements smaller than Γ under the linear ordering $\models$. Consult page 54 for its definition.

• $\bar{I}_\Gamma$–the reduced subset of I_Γ throwing away elements $\in I_\Gamma$ which are $\ll$ than some other element in I_Γ. Consult page 71 for its definition.

• $\bar{I}_\Gamma^{\gg}$–The subset of $\bar{I}_\Gamma$ collecting all the elements in I_Γ which are $\ll \Gamma$. Consult page 73 for its definition.

• j_i–the typical direct descendent index of i. The subscript i in j_i indicates j_i is a direct descendent of i.

• k_i–the subscripts in $\{1, 2, \cdots, n\}$ which corresponds to the indexes of type I exceptional classes with $e_{k_i} \cdot (C - \mathbf{M}(E)E) < 0$.

• Ξ_i–the $\mathbf{P}^1$ fibration over $Y(\Gamma_{e_i})$ representing the universal curves of the type I exceptional class e_i. The notation is used starting in section 3.

• $\tilde{\Xi}_i$–the relative minimal model of Ξ_i which has the $\mathbf{P}^1$ fiber bundle structure over $Y(\Gamma_{e_i})$. Please consult page 24, lemma 7 in subsection 3.1 for the construction.

• L–an line bundle over M with first Chern class $C \in H^{1,1}(M, \mathbf{Z})$. The bundle is assumed to be "sufficiently very ample" in this paper.

• M–an algebraic surface with irregularity $q = q(M)$ and geometric genus p_g.

• M_n–the $n-$th universal space associated with M. See section 2 for its construction.

• $\mathcal{M}_{C-\mathbf{M}(E)E}$ and $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i}$–the family moduli space associated with $C-\mathbf{M}(E)E$ and $C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i$, respectively. They can be viewed as the sub-schemes $Z(s_{canon})$ and $Z(s^{\circ}_{canon})$ of $\mathbf{P}(\mathbf{V}_{canon}) = \mathbf{P}(\mathbf{V}^{\circ}_{canon})$.

• $\mathbf{M}(E)E$–the multiple covering of E_i, $\sum_{1\leq i\leq n} m_i E_i$ with non-increasing singular multiplicities $0 < m_1 \leq m_2 \leq \cdots \leq m_n$.

• π–the projection map $f_n : M_{n+1} \mapsto M_n$ or its restriction to the various $\mathbf{P}^1$ fibrations $\Xi_i \mapsto Y(\Gamma_{e_i})$ or the union $\sum_{1\leq i} \Xi_{k_i} \mapsto Y(\Gamma)$.

• $\tilde{\pi}$–the projection morphism $\tilde{\Xi}_i \mapsto Y(\Gamma_{e_i})$ of the relative minimal model of Ξ_i.

• π_I–the projection map from M_n to $M^{|I|}$ determined by the index subset $I \subset \{1, 2, \cdots, n\}$.

• $\breve{\pi}_I$–the lifting of π_I to $M_n \mapsto M_{|I|}$.

• $\mathbf{Q}_{k_i}$ and $\mathcal{Q}_{k_i}$–the line bundle and the corresponding invertible sheaf appearing in the short exact sequence relating $\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}} \otimes \mathcal{E}_{-\mathbf{M}(E)E}\big)$ and $\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}}(e_{k_i})\big)$. Consult page 54 for more details.

• s_{canon}–the canonical section of $\pi^*_{\mathbf{P}(\mathbf{V}_{canon})}\mathbf{W}_{canon} \otimes \mathbf{H}$ induced by the bundle morphism $\mathbf{V}_{canon} \mapsto \mathbf{W}_{canon}$.

• s°_{canon}–the canonical section of $\pi^*_{\mathbf{P}(\mathbf{V}_{canon})}\mathbf{W}^\circ_{canon} \otimes \mathbf{H}$ determined by $\mathbf{V}^\circ_{canon} \mapsto \mathbf{W}^\circ_{canon}$.

• $\mathcal{S}_\Gamma$–the subset of M_n over which the type I exceptional cone $\mathcal{EC}_b(C - \mathbf{M}(E)E; Q)$ remains constant for $b \in \mathcal{S}_\Gamma$.

• $T(M)$–the connected component of the Picard variety of M parametrizing the line bundles with first Chern class C.

• $\mathbf{V}_{canon}$–the vector bundle over $M_n \times T(M)$ associated with the zero-th derived image sheaf $\mathcal{R}^0\pi_*\big(\mathcal{E}_C\big)$.

• $\mathbf{V}^\circ_{canon}$–the vector bundle over $M_n \times T(M)$ associated with the zero-th derived image sheaf $\mathcal{R}^0\pi_*\big(\mathcal{E}_C\big)$.

• $\mathbf{V}_{quot}$–the quotient bundle of $\mathbf{W}_{canon}$ whose associated locally free sheaf $\mathcal{V}_{quot}$ is constructed from the torsion free summand of a coherent sheaf $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}} \otimes \mathcal{E}_{C-\mathbf{M}(E)E}\big)$. Consult section 3.2 definition 4 for its definition, and proposition 5 on page 31 for its construction.

• $\tilde{\mathbf{V}}_{quot}$–a direct sum of vector bundles which is equivalent to $\mathbf{V}_{quot}$ in the K group. Consult 36, definition 4 for its definition.

• $\mathbf{W}_{canon}$ and $\mathcal{W}_{canon}$–$\mathbf{W}_{canon}$ is the canonical obstruction bundle associated with $C - \mathbf{M}(E)E$. $\mathcal{W}_{canon}$ is the locally free sheaf associated with $\mathbf{W}_{canon}$. Consult definition 5.3 of [Liu3] for its definition.

• $\mathbf{W}^\circ_{canon}$ and $\mathcal{W}^\circ_{canon}$–$\mathbf{W}^\circ_{canon}$ is the canonical obstruction bundle associated with $C - \mathbf{M}(E)E - \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i$. $\mathcal{W}^\circ_{canon}$ is the locally free sheaf associated with $\mathbf{W}^\circ_{canon}$. Consult section 5, right in front of lemma 6, of [Liu5] for its definition.

• $(\mathcal{X}/B)_n$–the $n-$th relative version of the universal space of $\mathcal{X} \mapsto B$. $\mathcal{X}/B$ is $\mathcal{X}$ with $''/B''$ to indicate that it has a fiber bundle structure over B.

• $Y(\Gamma)$, $\mathbf{Y}(\Gamma)$–$Y(\Gamma)$ is the closure of the admissible strata $Y_\Gamma \subset M_n$; $\mathbf{Y}(\Gamma)$ is the relative version $\subset (\mathcal{X}/B)_n$. Consult section 2 for some of its basic properties.

• Y_Γ, $\mathbf{Y}_\Gamma$–Y_Γ is the locally closed admissible stratum; $\mathbf{Y}_\Gamma$ is the relative version $\subset (\mathcal{X}/B)_n$. Consult section 2 for some of its properties.

• $\gg$–a partial ordering among pairs of the form $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$ with $\Gamma \in \Delta(n)$ which encodes the inclusion relationship of $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i} \times_{M_n} Y(\Gamma)$. Please consult page 57 for more details.

• $\sqsupset$–the partial ordering among pairs of the form $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$ with $\Gamma \in \Delta(n)$ which encodes the discrepancy of using $\mathbf{W}^\circ_{canon}$ and s°_{canon} to replace $\mathbf{W}_{canon}$ and s_{canon}. Please consult page 67 for more details.

• $\succ$–a partial ordering among $\Gamma \in \Delta(n)$ which encodes that the type I exceptional cone $\mathcal{C}_\Gamma$ gets larger under degenerations. Consult page 50 for more details.

• $\models$–The linear ordering introduced on $\Delta(n)$ and therefore on I_Γ and $\bar{I}_\Gamma$. Consult page 53 for more details.

• $\vdash$–the altered linear ordering on $\bar{I}_\Gamma$. Consult definition 17 for more details.

2 A Brief Review about the Admissible Graphs and Admissible Strata

In [Liu1] we had introduced the concepts of universal spaces (see also [V]), admissible graphs and the admissible stratification of the universal spaces M_n, $n \in \mathbf{N}$. For the convenience of the reader, we extract the basic facts about them in this section.

We review the construction of M_n and review the admissible graphs and the admissible stratification. Then we generalize it to a relative setting and discuss their basic properties and the relationship with type I exceptional classes.

Recall (consult section 3 on page 400 of [Liu1]) that the universal space M_n is constructed by an inductive procedure. Take $M_0 = pt$ and $M_1 = M$. Suppose that $M_0, M_1, M_2, \cdots, M_{k-1}$ has been constructed and there are natural projection maps $M_{k-1} \mapsto M_{k-2} \mapsto M_{k-3} \cdots \mapsto pt$, then define M_k to be the blowing up of the relative diagonal $\Delta_{M_{k-1}/M_{k-2}} : M_{k-1} \mapsto M_{k-1} \times_{M_{k-2}} M_{k-1}$. Then the natural projection map $f_{k-1} : M_k \mapsto M_{k-1}$ is a surjection. By mathematical induction, the universal spaces are defined for all $n \in \mathbf{N}$ and there are smooth surjective morphisms $f_k : M_{k+1} \mapsto M_k$ for all k. As usual the composite map $f_k \circ f_{k+1} \circ \cdots f_n : M_{n+1} \mapsto M_k$ will be denoted by $f_{n,k}$.

In [Liu1] we had introduced a concept called admissible graphs. The set of n-vertex admissible graphs, denoted by $adm(n)$, is the set of finite graphs with n vertexes and a finite number of arrowed one-edges which satisfy five axioms (from page 412-413 of [Liu1]).

Axiom 1: There is a $1-1$ correspondence between the vertexes of Γ and the positive integers smaller or equal to l. An association of this type is called a marking of the graph. More generally, one can mark the graph by any finite subset of $\mathbf{N}$. If $\mathbf{I}$ is the index set. The graph is called $\mathbf{I}$ admissible.

Axiom 2: The one-edges are oriented by arrows from the vertex marked by a smaller integer (called a direct ascendent) to the vertex marked by a larger integer (called a direct descendent).

Axiom 3: The only loops allowed in the graph are triangles formed by the three vertexes. Suppose $a < b < c$ are the three different vertexes, then b, c must be the direct descendents of a while a, b must be the direct ascendents of c. The vertexes a, b, c form a triangle.

Axiom 4: Any vertex can have at most two direct ascendents. When a vertex has exactly two direct ascendents, it and its two direct ascendents form a triangular loop.

Axiom 5: Suppose that two adjacent triangles share a common edge, then out of the four vertexes in the two triangles the ending vertex [3] of this common edge has exactly one direct descendent among the other three vertexes.

[3] at which the arrow points to

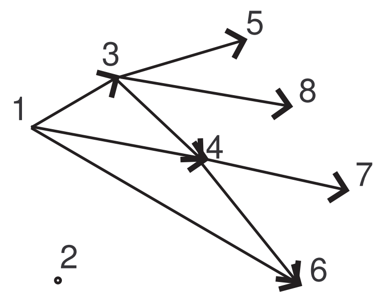

fig.1
An admissible graph with 8 vertexes.

The admissible graphs code the combinatorial patterns of the blowing ups. We usually denote a typical element in $adm(n)$ by Γ. The consistency of the above axioms were ensured geometrically in [Liu1] by constructing the corresponding admissible strata [4] $Y_\Gamma \neq \emptyset$ explicitly.

Definition 1 *Define the codimension of an admissible graph* $\Gamma \in adm(n)$, $codim_{\mathbf{C}}\Gamma$, *to be the number of one-edges in* Γ.

The graph in fig.1 illustrates an example of admissible graphs.

In the set $adm(n)$ there is a special element γ_n (or for simplicity skipping the subscript n and denote it by γ if it does not cause confusion) which consists of n free vertexes without one-edges. By construction $codim_{\mathbf{C}}\gamma_n = 0$ and γ_n is the only admissible graph in $adm(n)$ which has this special property.

The reason to introduce such a set $adm(n)$ is because that the universal space M_n can be stratified by the various admissible strata Y_Γ which have smooth closure $Y(\Gamma)$ in M_n,

Proposition 1 *The* $n-th$ *universal space* M_n *admits an admissible stratification* $M_n = \coprod_{\Gamma \in adm(n)} Y_\Gamma$ *into locally closed smooth subsets* Y_Γ *such that*

(i). The closure of Y_Γ *in* M_n, $\overline{Y_\Gamma} = Y(\Gamma)$ *is smooth of dimension* $dim_{\mathbf{C}}M_n - codim_{\mathbf{C}}\Gamma$ *in* M_n.

(ii). $Y(\Gamma)$ *can be expressed as a union of admissible strata* $Y_{\Gamma'}$, $Y(\Gamma) = \coprod_{\Gamma' < \Gamma} Y_{\Gamma'}$.

$\Gamma' < \Gamma$ indicates that $Y_{\Gamma'}$ appears in the compactification of Y_Γ into $Y(\Gamma)$ and is said to be a degeneration of the admissible graph Γ.

This proposition was proved in proposition 4.2.-4.3. in [Liu1]. In definition 4.8. of [Liu1] we had given a combinatorial characterization of the degenerations of the admissible graphs.

[4] Consult [Liu1] or below for the definition or construction of Y_Γ ($\mathbf{Y}_\Gamma$).

Remark 1 *The fibers of the fiber bundle map $f_n : M_{n+1} \mapsto M_n$ are all $n-$consecutive blowing ups from M. Therefore the space M_n can be interpreted as the " universal space" parametrizing all the ordered $n-$consecutive pointwise blowing ups from M.*

Let $M_n = \coprod_{\Gamma \in adm(n)} Y_\Gamma$ be the admissible stratification of M_n into locally closed strata. Then the stratum Y_{γ_n} is the only top dimensional stratum which parametrizes all the ordered distinct n points in M. Every distinct n points in M corresponds to n distinct pointwise blowing ups of M. The various Y_Γ, $\Gamma \neq \gamma_n$, parametrize those $n-$consecutive blowing ups whose blowing up centers may lie on the exceptional loci of the previous blowing ups.

Let $\mathcal{X} \mapsto B$ be a fiber bundle over a base B which is smooth of relative dimension two. We use the notation $\mathcal{X}/B$ to indicate that the space $\mathcal{X}$ has a structure of fiber bundle over B. We define $(\mathcal{X}/B)_0 = B$, $(\mathcal{X}/B)_1 = \mathcal{X}/B$. Suppose that $(\mathcal{X}/B)_1, (\mathcal{X}/B)_2, \cdots, (\mathcal{X}/B)_{k-1}$ have been defined and there are natural surjective projection maps $(\mathcal{X}/B)_{k-1} \mapsto (\mathcal{X}/B)_{k-2} \mapsto \cdots \mapsto B$, define $(\mathcal{X}/B)_k$ to be the blowing up of the relative diagonal $\Delta_{(\mathcal{X}/B)_{k-1}/(\mathcal{X}/B)_{k-2}} : (\mathcal{X}/B)_{k-1} \hookrightarrow (\mathcal{X}/B)_{k-1} \times_{(\mathcal{X}/B)_{k-2}} (\mathcal{X}/B)_{k-1}$, etc. Apparently $\mathbf{f}_{k-1} : (\mathcal{X}/B)_k \mapsto (\mathcal{X}/B)_{k-1}$ is surjective. By mathematical induction, $(\mathcal{X}/B)_n$ are constructed for all $n \in \mathbf{N}$ such that $\mathbf{f}_n : (\mathcal{X}/B)_{n+1} \mapsto (\mathcal{X}/B)_n$ are smooth and surjective.

For a given n, $\mathbf{f}_{n-1,0} : (\mathcal{X}/B)_n \mapsto B$ is the fiber bundle of the $n-$th universal spaces of the fibers.

We have the following lemma relating different relative universal space constructions,

Lemma 1 *For all $n \geq k$, we have the following identity*

$$(\mathcal{X}/B)_n/(\mathcal{X}/B)_k = \big((\mathcal{X}/B)_{k+1}/(\mathcal{X}/B)_k\big)_{n-k}.$$

Proof: This can be seen by noticing that for $n = k+1$, $(\mathcal{X}/B)_{k+1}/(\mathcal{X}/B)_k$ is a fiber bundle projection map. So we may take $\mathcal{X}' = (\mathcal{X}/B)_{k+1}$ and $B' = (\mathcal{X}/B)_k$ and $(\mathcal{X}/B)_{k+1}/(\mathcal{X}/B)_k = (\mathcal{X}'/B')_1$. Then by definition $(\mathcal{X}/B)_{k+2} \mapsto (\mathcal{X}/B)_k$ is the blowing up of the relative diagonal of $(\mathcal{X}/B)_{k+1} \times_{(\mathcal{X}/B)_k} (\mathcal{X}/B)_{k+1} = (\mathcal{X}'/B')_1 \times_{B'} (\mathcal{X}'/B')_1$. So we identify $(\mathcal{X}/B)_{k+2}/(\mathcal{X}/B)_k$ with $(\mathcal{X}'/B')_2$. By a simple induction argument and by comparing with the above relative construction of the universal spaces, we find $(\mathcal{X}/B)_n/(\mathcal{X}/B)_k = (\mathcal{X}'/B')_{n-k}$ for all $n \geq k$. Therefore we have the following identity,

$$(\mathcal{X}/B)_n/(\mathcal{X}/B)_k = \big((\mathcal{X}/B)_{k+1}/(\mathcal{X}/B)_k\big)_{n-k}.$$

$\square$

If we ignore the base space on the left hand side, we may rewrite the identity as $(\mathcal{X}/B)_n = \big((\mathcal{X}/B)_{k+1}/(\mathcal{X}/B)_k\big)_{n-k}$. By taking $B = pt$ and $\mathcal{X} = M$, we recover the important special case $M_n = (M_{k+1}/M_k)_{n-k}$.

Lemma 2 *For all $n \in \mathbf{N}$, there exists a canonical dominated birational map $(\mathcal{X}/B)_n \mapsto \times_B^n (\mathcal{X}/B)_1$ from the $n-th$ relative universal space $(\mathcal{X}/B)_n$ to the $n-th$ fiber products of $\mathcal{X}/B$.*

Proof: The assertion is apparently true for $n = 1$. Suppose that the birational map $(\mathcal{X}/B)_k \mapsto \times_B^k (\mathcal{X}/B)_1$ has been constructed, then

$$(\mathcal{X}/B)_{k+1} \mapsto (\mathcal{X}/B)_k \times_{(\mathcal{X}/B)_{k-1}} (\mathcal{X}/B)_k \mapsto \times_B^k (\mathcal{X}/B)_1 \times_{\times_B^{k-1}(\mathcal{X}/B)_1} \times_B^k (\mathcal{X}/B)_1 \cong \times_B^{k+1} (\mathcal{X}/B)_1$$

is a composition of dominated birational maps. By mathematical induction, the lemma is proved. □

The following proposition and its corollary are about the liftings of the projection maps to the corresponding (relative) universal spaces.

Proposition 2 *Let n be a positive integer and let $I \subset \{1, 2, \cdots, n\}$ be an index subset. Let $\pi_i : (\mathcal{X}/B)_n \mapsto (\mathcal{X}/B)$ be the composite projection map $(\mathcal{X}/B)_n \mapsto \times_B^n (\mathcal{X}/B) \mapsto \mathcal{X}/B$ to the $i-$th direct factor and let $\pi_I = \times_{i \in I} \pi_i : (\mathcal{X}/B)_n \mapsto \times_B^{|I|} (\mathcal{X}/B)$ be the projection to the fiber products of $|I|$ copies of $\mathcal{X}/B$ indexed by the subset I. Then there exists a natural lifting of the map π_I to $(\mathcal{X}/B)_n \mapsto (\mathcal{X}/B)_{|I|}$ which makes the following diagram commutative,*

$$\begin{array}{ccc} (\mathcal{X}/B)_n & \xrightarrow{\breve{\pi}_I} & (\mathcal{X}/B)_{|I|} \\ & \searrow & \downarrow \\ & & \times_B^{|I|} (\mathcal{X}/B) \end{array}$$

The lifted map $\breve{\pi}_I$ is smooth of relative dimension $2n - 2|I|$.

Notice that the lifted map $\breve{\pi}_I$ is usually different from the composite projection map $(\mathcal{X}/B)_n \mapsto (\mathcal{X}/B)_{n-1} \mapsto \cdots \mapsto (\mathcal{X}/B)_{|I|}$, which corresponds to the lifting of a very specific I.

Remark 2 *Following remark 1, the lifting map $\breve{\pi}_I$ can be interpreted as the "forgetful" map of forgetting all the blowing ups among the sequence of n-consecutive blowing ups marked by indexes in $\{1, 2, \cdots, n\} - I$.*

Proof of the proposition: When $n = 1$, the statement holds trivially.

Let $2 \leq n \in \mathbf{N}$ be a positive integer such that the statement of the proposition is known to be true for $n-1$, for all the $\mathcal{X} \mapsto B$ pairs. We would like to show that it holds for n as well. Suppose that $I = \{1, 2, \cdots, n\}$, the statement holds since the lifted map $(\mathcal{X}/B)_n \mapsto (\mathcal{X}/B)_{|I|}$ is the identity map. So let us assume that $|I| < n$. By lemma 1 we may rewrite $(\mathcal{X}/B)_n$ as $((\mathcal{X}/B)_2/(\mathcal{X}/B)_1)_{n-1}$.

♢ We separate into two cases. (i). $1 \notin I$. (ii). $1 \in I$.

In the first case, the projection to the second factor $\pi_2 : (\mathcal{X}/B)_2 \mapsto (\mathcal{X}/B)_1$ induces a morphism $(\mathcal{X}/B)_2/(\mathcal{X}/B)_1 \mapsto (\mathcal{X}/B)_1/pt$. This morphism induces an morphism on the relative $n-1-$th universal spaces $((\mathcal{X}/B)_2/(\mathcal{X}/B)_1)_{n-1} \mapsto ((\mathcal{X}/B)_1/pt)_{n-1} \cong (\mathcal{X}/B)_{n-1}$. By the assumption $1 \notin I$ we know that $I \subset \{2, 3, \cdots, n\}$. Define a new index set $I_{-1} \subset \{1, 2, \cdots, n-1\}$ by subtracting 1 from all the elements of I. Then by the induction hypothesis, $\pi_{I_{-1}}$:

$(\mathcal{X}/B)_{n-1} \mapsto \times_B^{|I_{-1}|}(\mathcal{X}/B)$ can be lifted to the smooth surjective morphism $\check{\pi}_{I_{-1}} : (\mathcal{X}/B)_{n-1} \mapsto (\mathcal{X}/B)_{|I_{-1}|}$. By composing with

$$(\mathcal{X}/B)_n \cong ((\mathcal{X}/B)_2/(\mathcal{X}/B)_1)_{n-1} \mapsto ((\mathcal{X}/B)_1/pt)_{n-1} \cong (\mathcal{X}/B)_{n-1},$$

we get the desired lifting from $(\mathcal{X}/B)_n \cong ((\mathcal{X}/B)_2/(\mathcal{X}/B))_{n-1} \mapsto \times_B^{|I_{-1}|}(\mathcal{X}/B) = \times_B^{|I|}(\mathcal{X}/B)$ to $(\mathcal{X}/B)_n \mapsto (\mathcal{X}/B)_{|I_{-1}|} = (\mathcal{X}/B)_{|I|}$. The lifted surjective map is apparently smooth because all the composite factors of maps are smooth and surjective.

In the second case when $1 \in I$, we construct an new index set I'_{-1} by subtracting all elements in $I - \{1\}$ by 1, then $I'_{-1} \subset \{1, 2, \cdots, n-1\}$ and we have $|I'_{-1}| = |I| - 1$. We define a new fiber bundle $\mathcal{X}' \mapsto B'$ smooth of relative dimension two by setting $\mathcal{X}' = (\mathcal{X}/B)_2, B' = (\mathcal{X}/B)_1$. Then we may rewrite $(\mathcal{X}/B)_n$ as $((\mathcal{X}/B)_2/(\mathcal{X}/B)_1)_{n-1} = (\mathcal{X}'/B')_{n-1}$. By the induction hypothesis, the map $\pi'_{I'_{-1}} : (\mathcal{X}'/B')_{n-1} \mapsto \times_{B'}^{|I'_{-1}|}(\mathcal{X}'/B')$ can be lifted to $\check{\pi}'_{I'_{-1}} : (\mathcal{X}'/B')_{n-1} \mapsto (\mathcal{X}'/B')_{|I'_{-1}|}$.

On the other hand $|I'_{-1}| = |I| - 1$, we realize by using lemma 1 again that

$$(\mathcal{X}'/B')_{|I'_{-1}|} = (\mathcal{X}'/B')_{|I|-1} = ((\mathcal{X}/B)_2/(\mathcal{X}/B)_1)_{|I|-1} \cong (\mathcal{X}/B)_{|I|-1+1} = (\mathcal{X}/B)_{|I|}.$$

Then the lifted maps from $(\mathcal{X}/B)_n = (\mathcal{X}'/B')_{n-1}$ to $(\mathcal{X}'/B')_{|I|-1} = (\mathcal{X}/B)_{|I|}$ is the desired lifting map $\check{\pi}_I$. By inductive hypothesis, it is smooth and surjective.

Based on mathematical induction, the existence of the lifting is proved. Finally the assertion about the relative dimension $2n - 2|I|$ is by a direct comparison of the dimensions of the source and the target. $\Box$

Corollary 1 *Let $\pi_i : M_n \mapsto M, 1 \le i \le n$ denote the projection to the $i-th$ copy of M. Let $\pi_I = \times_{i \in I}\pi_i : M_n \mapsto M^{|I|}$ is the projection to the $|I|$-Cartesian product of M indexed by I.*

Then there exists a unique surjective and smooth lifting $\check{\pi}_I : M_n \mapsto M_{|I|}$ which makes the following diagram commutative,

$$\begin{array}{ccc} M_n & \xrightarrow{\check{\pi}_I} & M_{|I|} \\ & \searrow & \downarrow \\ & & M^{|I|} \end{array}$$

The map $\check{\pi}_I$ is of relative dimension $2n - 2|I|$.

Proof: By taking $M = \mathcal{X} \mapsto B = pt$ in the proposition 2, the corollary is a direct consequence of proposition 2. $\Box$

Given an $n \in \mathbf{N}$, one may prove inductively (see lemma 3.1 and proposition 3.1 on pages 401-402 of [Liu1]) that the birational map $M_{n+1} \mapsto M \times M_n$ can be factorized into n codimension-two blowing ups along the cross sections of the intermediate fiber bundles $f^*_{n-1,i}M_{i+1} \mapsto M_n$ induced by the relative diagonals

$M_{i+1} \hookrightarrow M_{i+1} \times_{M_i} M_{i+1}$. As a consequence, M_n can be blown up from M^n by $\frac{n(n-1)}{2}$ consecutive codimension-two blowing ups along the partial diagonals. Our convention in this paper is that $E_{a;b}, 1 \leq a < b \leq n$ denote the (pull-back of) the exceptional divisor blown up from the strict transforms of the $(a,b)-$th partial diagonals. Under this convention, the n distinct exceptional divisors, $E_{1;n+1}$, $E_{2;n+1}$, $\cdots$, $E_{n;n+1}$ of the fiber bundle $f_n : M_{n+1} \mapsto M_n$ are denoted by $E_1, E_2, \cdots, E_n$ respectively. They will play a special role in this paper.

Let Γ be an admissible graph $\in adm(n)$, then one may attach n distinct type I exceptional classes e_i, $1 \leq i \leq n$, to Γ. Given an index i, with $1 \leq i \leq n$, let the indexes j_i run through all the direct descendents of i in Γ. Then $e_i = E_i - \sum_{j_i} E_{j_i}$ is the $i-$th type I exceptional class attached to Γ. We set $e_i \cdot e_j$ to be the fiberwise intersection number of the classes e_i and e_j.

The following proposition will be used frequently in this paper,

Proposition 3 *For $1 \leq i \leq n$ let J_i be an index subset of $\{1, 2, \cdots, n\}$ satisfying $inf(J_i) > i$. Let $e_1, e_2, \cdots, e_n$ be n divisor classes of the form $E_i - \sum_{j \in J_i} E_j$. Suppose that $e_1, e_2, \cdots, e_n$ satisfy the condition $e_a \cdot e_b \geq 0$ for all $a \neq b$. Then there exists an admissible graph $\Gamma \in adm(n)$ such that $e_1, e_2, e_3, \cdots, e_n$ are the type I exceptional classes associated with Γ.*

In other words, the locus $Y(\Gamma) \subset M_n$ is the locus of co-existence over which $e_1, e_2, \cdots, e_n$ become effective. Please see fig.3 on page 16 for an example that Γ is recovered from the fan-like graphs associated with these e_i.

Proof: The proposition is proved by an induction argument on n. The base case $n = 1$ is trivial. Suppose that the proposition has been proved for $n - 1$, we would like to prove the existence of such a $\Gamma \in adm(n)$ for n. Given n vertexes, construct the graph Γ by the following rule: Given an $i \leq n$, connect an oriented edge from i to any $j > i$ if $j \in J_i$, i.e. if the term $-E_j$ appears in the class e_i. We show that Γ is an admissible graph, i.e. it satisfies the five axioms characterizing admissible graphs.

Firstly we shift all the indexes by -1 temporally and denote the new graph marked by the shifted indexes $\{0, 1, 2, \cdots, n-1\}$ as $\tilde{\Gamma}$. It is easy to see that Γ is admissible with respect to $\{1, 2, \cdots, n\}$ iff $\tilde{\Gamma}$ is admissible [5] with respect to the shifted index set $\{0, 1, \cdots, n-1\}$. Define $\phi : \mathbf{Z} \mapsto \mathbf{Z}$ by the formula $\phi(i) = i - 1$. Define $\tilde{J}_i = \phi(J_{i+1})$ for $i \in \{0, 1, 2, \cdots, n-1\}$. Likewise define $\tilde{E}_i = E_{i+1}$. Define accordingly $\tilde{e}_i = \tilde{E}_i - \sum_{j \in \tilde{J}_i} \tilde{E}_j$. It is clear that $\tilde{e}_i = e_{i+1}$ for $0 \leq i \leq n-1$ and their mutual intersection pairings are still non-negative.

Thus by our induction hypothesis, the classes [6] $\tilde{e}_1, \tilde{e}_2, \cdots, \tilde{e}_{n-1}$ satisfy $\tilde{e}_i \cdot \tilde{e}_j = e_{i+1} \cdot e_{j+1} \geq 0$ for $i \neq j$. So there exists an admissible graph $\Gamma' \in adm(n-1)$ such that $\tilde{e}_1, \cdots, \tilde{e}_{n-1}$ are the type I exceptional classes associated to Γ'. And by the inductive construction Γ' is constructed by the datum of $\tilde{e}_1, \cdots, \tilde{e}_{n-1}$ the same way we construct Γ (and $\tilde{\Gamma}$). So it is clear that Γ' is a sub-graph of $\tilde{\Gamma}$ by removing the $0-$th vertex and all the arrowed edges starting from it. That

[5] relaxing the constraint on the index set.

[6] Notice that the class $\tilde{e}_0$ is excluded here.

is because $\tilde{e}_0$ is not used in constructing Γ'. Our inductive assumption implies that this sub-graph Γ' of $\tilde{\Gamma}$ is admissible (with respect to $\{1, 2, \cdots, n-1\}$).

Our final task is to show the admissibility of the whole $\tilde{\Gamma}$. Among the five axioms which characterize admissible graphs, the axiom 1. and axiom 2. are satisfied trivially by construction.

To show that axioms 3., 4., 5., are satisfied, we prove by contradiction.

$\diamondsuit$ About axiom 3.: if there is a polygonal loop in $\tilde{\Gamma}$ which is not a triangle, we may choose a polygonal loop of vertexes involving the least number of vertexes/edges. Firstly, this 'exotic loop' cannot locate completely within Γ' because Γ' has been known to be admissible, so axiom 3 for Γ' rules out this possibility. So the vertex marked by 0 must be within this loop. Consider the vertex v marked by the largest integer q along the loop. It must be located in Γ' because $q \neq 0$. On the other hand, the loop passes through v means that there are two edges ending at v (because the arrows of the edges always point to vertexes marked by the larger integers).

If both direct ascendents v_1, v_2 of v are not marked by 0, they are vertexes in Γ' as well and by the admissibility of Γ', axiom 4. for Γ' implies that v, v_1, v_2 form a triangle. This implies that we can shorten the loop by replacing the oriented edges $\overrightarrow{v_1 v}$ and $\overrightarrow{v_2 v}$ by the single edge $\overrightarrow{v_1 v_2}$ (or $\overrightarrow{v_2 v_1}$, depending on which vertex is marked by a larger integer). This violates the assumption that the loop involves the least number of vertexes/edges!

If one of the two direct ascendents v_1, v_2 of v, say v_1, is marked by 0, assume that v_2 is marked by p, with $1 \leq p \leq n-1$. Then $q \in \tilde{J}_p$ and $q \in \tilde{J}_0$ simultaneously.

On the other hand, we have assumed that $\tilde{e}_0 \cdot \tilde{e}_p \geq 0$. As $(-E_q)^2$ contributes -1 to the intersection number, there must be a positive counter-term which makes the intersection number non-negative. This only occurs when $p \in \tilde{J}_0$ and $(E_p) \cdot (-E_q) = 1$ contributes positively to the sum. But this implies that the edges linking v_1, v_2, v already form a triangle and it is not a polygonal loop.

In any case, the axiom 3. holds for $\tilde{\Gamma}$.

$\diamondsuit$ About axiom 4., suppose that a vertex in $\tilde{\Gamma}$ has more than two direct ascendents in $\tilde{\Gamma}$. Then we randomly pick three of them, say v_1, v_2, v_3 (marked by $p_1 < p_2 < p_3$, respectively), and derive a contradiction.

Because $q \in \tilde{J}_{p_1} \cap \tilde{J}_{p_2} \cap \tilde{J}_{p_3}$, by the same argument in checking axiom 3., $\tilde{e}_{p_1} \cdot \tilde{e}_{p_2} \geq 0$, $\tilde{e}_{p_1} \cdot \tilde{e}_{p_3} \geq 0$, and $\tilde{e}_{p_2} \cdot \tilde{e}_{p_3} \geq 0$ force v_2, v_3 to be the direct descendents of v_1 while v_3 is forced to be a direct descendent of v_2. Having made such an observation, we recalculate $\tilde{e}_{p_1} \cdot \tilde{e}_{p_2}$ again. Because $p_2 \in \tilde{J}_{p_1}$, but $\{p_3, q\} \subset \tilde{J}_{p_1} \cap \tilde{J}_{p_2}$. This implies that

$$0 \leq \tilde{e}_{p_1} \cdot \tilde{e}_{p_2} \leq (-E_{p_2}) \cdot E_{p_2} + (-E_{p_3})^2 + (-E_q)^2 = -1 < 0.$$

This is absurd! Thus v can have at most two direct descendents in $\tilde{\Gamma}$.

When v has exactly two direct ascendents v_1, v_2 marked by $p_1 < p_2$, $\tilde{e}_{p_1} \cdot \tilde{p_2} \geq 0$ and $q \in \tilde{J}_{p_1} \cap \tilde{J}_{p_2}$ imply that v_2 is also a direct descendent of v_1 and v_1, v_2, v form a triangle in $\tilde{\Gamma}$. So the axiom 4. is satisfied.

$\diamondsuit$ Finally about axiom 5.: Suppose that two adjacent triangles are sharing a common one-edge. Let v_1 and v be the starting and the ending vertexes of the one-edge. Suppose that v_2, v_3 are the other two vertexes in these two triangles, let p_1, p_2, p_3 and q mark the vertexes v_1, v_2, v_3 and v, respectively. Because v are in both of the triangles, there must be two different one-edges linking v and v_2, v and v_3, respectively. There are three exclusive possibilities. Either

(i). both v_2, v_3 are the direct ascendents of v.

(ii). One of them, say v_2, is the direct ascendent of v and the other vertex v_3 is the direct descendent of v.

(iii). Both of v_2 and v_3 are direct descendents of v.

If the possibility (i) holds, then v has at least three direct ascendents in $\tilde{\Gamma}$ and this violates axiom 4 for $\tilde{\Gamma}$, which we have proved already.

If the possibility (iii). holds, then we argue v_2, v_3 are the direct descendent of v_1 as well.

By our assumption on the existence of adjacent triangles, there must be one-edges between v_1, v_2 and v_1, v_3. But v is already known to be a direct descendent of v_1. If v_2, v_3 are known to be the direct descendents of v, then v_1 cannot be a direct descendent of either v_2 or v_3. So the arrows of the oriented one-edges must go from v_1 to v_2 and v_3, respectively. Then v_2, v_3 must be the direct descendents of v_1.

Now the vertex v_1 has at least three direct descendents v_2, v_3 and v while the vertex v has at least two direct descendents v_2, v_3.

This implies that $\{p_2, p_3\} \subset \tilde{J}_{p_1} \cap \tilde{J}_q$ and a direct calculation on $\tilde{e}_{p_1} \cdot \tilde{e}_q$ shows that

$$0 \leq \tilde{e}_{p_1} \cdot \tilde{e}_q \leq (-\tilde{E}_q) \cdot (\tilde{E}_q) + (-\tilde{E}_{p_2})^2 + (-\tilde{E}_{p_3})^2 = 1 - 1 - 1 = -1 < 0.$$

A contradiction to our assumption! So the only possibility is (ii) and exactly one of v_2 or v_3 can be the direct descendent of v.

As $\tilde{\Gamma}$ satisfies all five axioms, it is admissible with respect to the marking $\{0, 1, 2, \cdots, n-1\}$. Thus the original Γ is admissible with respect to $\{1, 2, \cdots, n\}$. So $\Gamma \in adm(n)$. $\Box$

Conversely for all $\Gamma \in adm(n)$ the smooth and closed set $Y(\Gamma) \subset M_n$ can be identified to be the transversal intersection $\cap_{1 \leq i \leq n} Y(\Gamma_{e_i})$, where Γ_{e_i} is the fan-like admissible graph $\in adm(n)$ such that (i). the vertex marked by i is the only direct ascendent among the n vertexes. (ii). The direct descendents of the vertex marked by i are the direct descendent indexes of i in Γ. Thus $Y(\Gamma)$ can be viewed as the locus over which all the type I exceptional classes $e_i, 1 \leq i \leq n$ are simultaneously effective along the fibers of $M_{n+1} \mapsto M_n$. The result has been proved in proposition 4.7. of [Liu1], using slightly different terminologies in terms of pseudo-holomorphic curves.

The graphs in fig.2 are the fan-like sub-graphs from fig.1 on page 8.

For the ease of the reader with algebraic geometric background, we give an alternative relative construction of $Y(\Gamma)$ which makes the above property manifest.

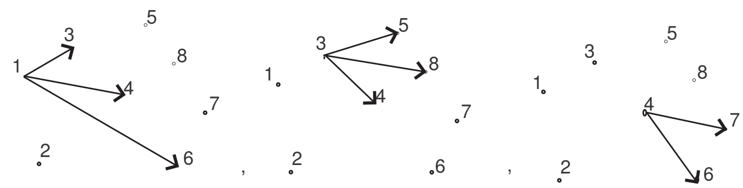

fig.2
The fan-like subgraphs Γ_{e_1}, Γ_{e_3} and Γ_{e_4} of the admissible graph Γ in fig.1.

Let $(\mathcal{X}/B)_n$ be the $n-$th relative universal space over B. The fiber of $(\mathcal{X}/B)_n$ over $b \in B$ is nothing but the $n-$th universal space of the fiber of $\mathcal{X}/B$ above b.

By lemma 1 we have the following canonical isomorphism $\big((\mathcal{X}/B)_{i+1}/(\mathcal{X}/B)_i\big)_{n-i} = (\mathcal{X}/B)_n$ for each $i \leq n$. Thus we may set $B'_i = (\mathcal{X}/B)_i$ for all i and there is a surjection $(\mathcal{X}/B)_n \mapsto \times^{n-i}_{B'_i}\big((\mathcal{X}/B)_{i+1}/B'_i\big)$ to the $n-i-$fold fiber product of $(\mathcal{X}/B)_{i+1}$ over B'_i.

Parallel to the absolute version, for all $\Gamma \in adm(n)$ we may define the relative admissible strata $\mathbf{Y}_\Gamma$ (or the closure $\mathbf{Y}(\Gamma)$) [7] to be the union of fiberwise Y_Γ (or $Y(\Gamma)$) over[8] B.

By the previous inductive construction on the relative universal spaces, $(\mathcal{X}/B)_{i+1}/B'_i$ is the blowing up of the relative diagonal $\Delta_{B'_i/B'_{i-1}} : B'_i \mapsto B'_i \times_{B'_{i-1}} B'_i$. Let $D_i \mapsto B'_i$ with $D_i \subset (\mathcal{X}/B)_{i+1}$ denote the blown up exceptional divisor in $(\mathcal{X}/B)_{i+1}$, which has a structure of $\mathbf{P}^1$ bundle over B'_i. Let J_i denote the set of direct descendent indexes of i in Γ. Let $s = |J_i|$ be the cardinality of J_i, the number of direct descendents of i in Γ_{e_i}. The number s is also equal to $codim_{\mathbf{C}}\Gamma_{e_i}$.

The inclusion of the fiber bundle $D_i/B'_i \hookrightarrow (\mathcal{X}/B)_{i+1}/B'_i$ induces the canonical map on the $|J_i|-$th relative universal spaces,

$$(D_i/B'_i)_{|J_i|} \hookrightarrow ((\mathcal{X}/B)_{i+1}/B'_i)_{|J_i|} \cong (\mathcal{X}/B)_{|J_i|+i}.$$

However, the fiber bundle $D_i \mapsto B'_i$ is smooth of relative dimension one. So by a direct check we find (using the fact the codimension one blowing ups are trivial) $(D_i/B'_i)_s \cong \times^s_{B'_i} D_i$, the s-fold fiber product of D_i over B'_i. On the other hand, proposition 2 implies that for $\mathcal{X}'/B'_i = (\mathcal{X}/B)_{i+1}/B'_i$, $\pi_{J_i} : (\mathcal{X}'/B'_i)_{n-i} \mapsto \times^{|J_i|}_{B'_i}(\mathcal{X}'/B'_i)$ can be lifted to a smooth and surjective map $\breve{\pi}_{J_i} : (\mathcal{X}'/B'_i)_{n-i} \mapsto (\mathcal{X}'/B'_i)_{|J_i|}$. Moreover, the isomorphism $(\mathcal{X}/B)_n \stackrel{\psi_{i,n}}{\longrightarrow} ((\mathcal{X}/B)_{i+1}/B'_i)_{n-i}$ allows

[7] Here we use bold $\mathbf{Y}$ to denote the relative versions of Y_Γ or $Y(\Gamma)$

[8] See remark 3 for an outline of an alternative inductive definition after proposition 4

us to view the $\psi_{i,n}^{-1}$ pre-image of the relative [9] $\mathbf{Y}_{\gamma_{n-i}}$ over B'_i as a subset of $(\mathcal{X}/B)_n$.

We have the following characterization of $\mathbf{Y}(\Gamma_{e_i})$ and $\mathbf{Y}_{\Gamma_{e_i}}$,

Lemma 3 *The closed subspace $\mathbf{Y}(\Gamma_{e_i}) \subset (\mathcal{X}/B)_n$, smooth of codimension $codim_{\mathbf{C}}\Gamma_{e_i}$ in $(\mathcal{X}/B)_n$, is the pre-image of $\times^s_{B'_i} D_i \subset ((\mathcal{X}/B)_{i+1}/B'_i)_{|J_i|}$ under $\check{\pi}_{J_i}^{-1}$. Likewise the locally closed subset $\mathbf{Y}_{\Gamma_{e_i}} \subset \mathbf{Y}(\Gamma_{e_i})$ can be identified with $\mathbf{Y}(\Gamma_{e_i}) \cap \psi_{i,n}^{-1}(\mathbf{Y}_{\gamma_{n-i}})$.*

Proof: For all $b \in B$, by remark 1 we know that $(\mathcal{X}_b)_n$ parametrizes all the ordered $n-$consecutive pointwise blowing ups of $\mathcal{X}_b$. Given the fan-like admissible graph Γ_{e_i}, an ordered $n-$consecutive blowing ups from $\mathcal{X}_b$ lies in the fiberwise $Y(\Gamma_{e_i})$ of $(\mathcal{X}/B)_n \mapsto B$ above $b \in B$ iff all the $k_1, k_2, \cdots, k_s-$th blown up points, $k_l \in J_i$, $1 \leq l \leq s = codim_{\mathbf{C}}\Gamma_{e_i} = |J_i|$, lie above the exceptional $\mathbf{P}^1$ of the $i-$th blown up point. Over the relative $i-$th universal space $(\mathcal{X}/B)_i$ which parametrizes the first $i-$th blowing ups in the B family, the union of the $i-$th exceptional $\mathbf{P}^1$ forms a fiber bundle, which is nothing but $D_i \mapsto (\mathcal{X}/B)_i$ introduced above. On the one hand all the $k_1, k_2, \cdots, k_s-$th blown up points are allowed to move on the fibers of D_i freely. This implies that $\mathbf{Y}(\Gamma_{e_i}) \xrightarrow{\check{\pi}_{J_i}} \times^s_{(\mathcal{X}/B)_i} D_i$ must be surjective. On the other hand, for all $j > i$ which are not the direct descendent indexes of i, the $j-$th blowing up centers within the $n-$consecutive pointwise blowing ups are not restricted at all. Therefore $\mathbf{Y}(\Gamma_{e_i})$ can be identified with $\check{\pi}_{J_i}^{-1}(\times^s_{(\mathcal{X}/B)_i} D_i)$.

Inside this smooth space $\mathbf{Y}(\Gamma_{e_i})$, the sub-locus $\mathbf{Y}_{\Gamma_{e_i}}$ corresponds to the set of all the $n-$consecutive blowing ups from $\mathcal{X}_b$ above all $b \in B$ such that it is in the fiberwise $Y(\Gamma_{e_i})$ and none of blowing ups marked by $\{i+1, i+2, \cdots n\}$ lie above the exceptional loci of one another. Thus the space $\mathbf{Y}_{\Gamma_{e_i}}$ must map into the relative $\mathbf{Y}_{\gamma_{n-i}}$ over $B'_i = (\mathcal{X}/B)_i$ under $\psi_{i,n} : (\mathcal{X}/B)_n \mapsto \big((\mathcal{X}/B)_{i+1}/(\mathcal{X}/B)_i\big)_{n-i}$, as the space $\mathbf{Y}_{\gamma_{n-i}}$ parametrizes all the disjoint last $n-i-$th consecutive blowing ups of the family $(\mathcal{X}/B)_{i+1}/B'_i$ of algebraic surfaces. Thus $\mathbf{Y}_{\Gamma_{e_i}} = \mathbf{Y}(\Gamma_{e_i}) \cap \psi_{i,n}^{-1}(\mathbf{Y}(\gamma_{n-i}))$ is locally closed. $\square$

By a direct calculation, $codim_{\mathbf{C}}\Gamma_{e_i}$, being the number of direct descendents of Γ_{e_i}, is also equal to the negation of $d_{GT}(e_i) = \frac{e_i^2 - c_1(\mathbf{K}_{M_{n+1}/M_n})\cdot e_i}{2}$. This implies that the existence locus $\subset (\mathcal{X}/B)_n$ of the fiberwise class $e_i \in \mathcal{A}.((\mathcal{X}/B)_{n+1}/(\mathcal{X}/B)_n)$ (over which e_i becomes effective along the fibers) is smooth of the expected dimension $dim_{\mathbf{C}}(\mathcal{X}/B)_n) + d_{GT}(e_i)$. Moreover $\mathbf{Y}_{\Gamma_{e_i}}$ is the locus over which the type I class e_i is effective and irreducible/smooth in the fibers of $(\mathcal{X}/B)_{n+1} \mapsto (\mathcal{X}/B)_n$.

The following proposition characterizes $\mathbf{Y}(\Gamma)$ in terms of the fan-like graphs Γ_{e_i} and can be viewed as the converse of proposition 3,

Proposition 4 *Let $\Gamma \in adm(n)$ be an $n-$vertex admissible graph. The smooth and closed subspace $\mathbf{Y}(\Gamma) \subset (\mathcal{X}/B)_n$ of $codim_{\mathbf{C}}\Gamma$ can be identified with the*

[9] From the subscript of γ_{n-i}, one should be able to distinguish $\mathbf{Y}(\gamma_n) \subset (\mathcal{X}/B)_n$ over B and $\mathbf{Y}_{\gamma_{n-i}} \subset \big((\mathcal{X}/B)_{i+1}/B'_i\big)_{n-i}$ over $B'_i = (\mathcal{X}/B)_i$.

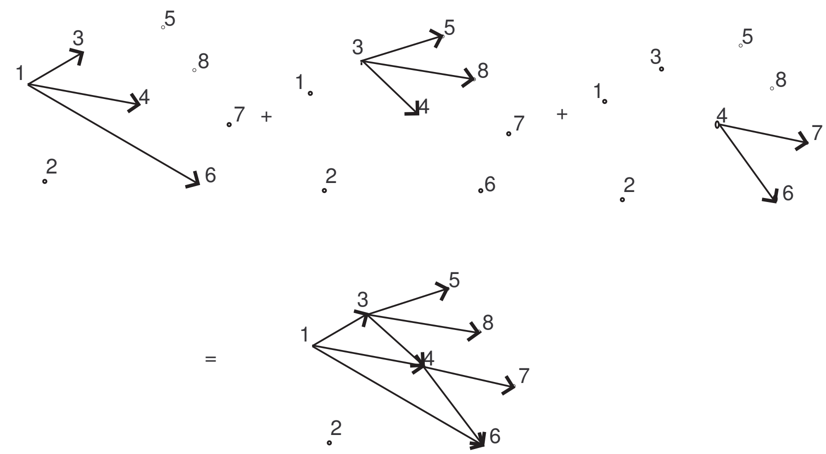

fig.3

When we fuse the admissible graphs Γ_{e_1}, Γ_{e_3}, Γ_{e_4} together, we recover the original Γ

regular intersection $\cap_{1\leq i\leq n}\mathbf{Y}(\Gamma_{e_i})$. *Likewise the locally closed smooth subspace* $\mathbf{Y}_\Gamma \subset \mathbf{Y}(\Gamma)$ *is equal to the intersection* $\cap_{1\leq i\leq n}\mathbf{Y}_{\Gamma_{e_i}}$.

Proof:

$\diamondsuit$ Auxiliary Statement: Let $K \subset \{1, 2, \cdots, n\}$ be an index subset. We claim that the restricted lifted mapping $\breve{\pi}_K : \mathbf{Y}(\Gamma) \mapsto (\mathcal{X}/B)_{|K|}$ (or its restriction to the subspace $\mathbf{Y}(\Gamma)$) maps smoothly onto $\mathbf{Y}(\Gamma_K) \subset (\mathcal{X}/B)_{|K|}$ (or $\mathbf{Y}(\Gamma_K)$, for an admissible $\Gamma_K \in adm(|K|)$ characterized as the following: Firstly, the map $\breve{\pi}_K$ induces an ordering preserving bijection $\phi_K : K \mapsto [1, 2, \cdots, |K|]$ between index sets. Then the type I exceptional cycle classes $e_i, 1 \leq i \leq n$ along the fibers of $(\mathcal{X}/B)_{n+1} \times_{(\mathcal{X}/B)_n} \mathbf{Y}(\Gamma) \mapsto \mathbf{Y}(\Gamma)$ are pushed-forward to fiberwise cycle classes of $(\mathcal{X}/B)_{|K|+1} \mapsto (\mathcal{X}/B)_{|K|}$ by the following rule: The $e_i \mapsto 0, i \notin K$; but $e_i \mapsto e^K_{\phi_K(i)}$ for $i \in K$. Those $e^K_{\phi_K(i)}$ are constructed from $e_i, i \in K$, by the following substitutions: $\mathcal{A}.((\mathcal{X}/B)_{n+1}) \ni E_a \mapsto E_{\phi_K(a)} \in \mathcal{A}.((\mathcal{X}/B)_{|K|+1})$, $a \in K$, and $E_a \mapsto 0$, $a \notin K$. By a simple calculation based on the substitution rules we find that $e^K_i \cdot e^K_j \geq e_{\phi^{-1}_K(i)} \cdot e_{\phi^{-1}_K(j)} \geq 0$ for all $i \neq j$, $i, j \in [1, \cdots, |K|]$. Then the desired $\Gamma_K \in adm(|K|)$ is constructed from the collection of classes $e^K_j, 1 \leq j \leq |K|$, by applying proposition 3.

We prove the proposition along with the auxiliary statement $\diamondsuit$ based on induction arguments on n.

For $n = 1$, there is only one admissible graph $\Gamma = \gamma_1 \in adm(1)$ and the proofs of both the statements are trivial. Suppose that the statements have been proved for all the pairs $\mathcal{X} \mapsto B$ for the natural numbers $\leq n$, we prove them for $n + 1$.

Let $\Gamma \in adm(n + 1)$ be an $n + 1$-vertex admissible graph. As in the proof of proposition 3, we subtract all the indexes by -1 and denote the resulting graph by $\tilde{\Gamma}$. Then as before the subgraph Γ', marked by $\{1, 2, \cdots, n\}$, is an admissible graph $\in adm(n)$ in the usual sense.

By lemma 1 we have $(\mathcal{X}/B)_{n+1} = \big((\mathcal{X}/B)_2/(\mathcal{X}/B)_1\big)_n$. Over the base space $\mathcal{X}/B$ the locus $\mathbf{Y}(\Gamma')$ defines a closed subset of the relative $n-$th universal space of $(\mathcal{X}/B)_2 \mapsto (\mathcal{X}/B)_1$, $(\mathcal{X}/B)_{n+1}/(\mathcal{X}/B)_1$. By induction hypothesis, we may assume $\mathbf{Y}(\Gamma') = \cap_{1\leq i\leq n}\mathbf{Y}(\Gamma'_{e'_i})$ to be a regular intersection. The class $e'_i/\Gamma'_{e'_i}$ are the type I exceptional classes/fan-like admissible graphs associated with the vertexes of Γ'. On the other hand, Γ' is the subgraph of $\tilde{\Gamma}$ removing the $0-$th vertex and all the arrowed one-edges starting from '0'. So $e'_i = \tilde{e}_i$ and $\Gamma'_{e'_i} = \tilde{\Gamma}_{\tilde{e}_i}$ for $1 \leq i \leq n$. As before we define J_0 to be the set of all direct descendent indexes of $'0'$ in $\tilde{\Gamma}$. By lemma 3, $\mathbf{Y}(\tilde{\Gamma}_{\tilde{e}_0}) \subset \big((\mathcal{X}/B)_2/(\mathcal{X}/B)_1\big)_n = (\mathcal{X}/B)_{n+1}$ is smooth of codimension $codim_{\mathbf{C}}\tilde{\Gamma}_{\tilde{e}_0}$, the pre-image of $\times_{(\mathcal{X}/B)_1}^{codim_{\mathbf{C}}\tilde{\Gamma}_{\tilde{e}_0}} D_0$ under $\check{\pi}_{J_0}^{-1}$.

Consider the admissible graph $\Gamma'_{J_0} = \Gamma_{J_0} \in adm(|J_0|)$, constructed by the recipe at the beginning of our proof. We claim that it must be a finite union of linear chains [10]. This is equivalent to say that all the type I exceptional classes associated to Γ'_{J_0} are either -1 or -2 classes. If there is a $-k$ class (with $k > 2$) among the type I exceptional classes of Γ'_{J_0}, then there is an index a with more than one direct descendent in Γ'_{J_0}. By the construction of $\Gamma'_{J_0} = \Gamma_{J_0}$ from Γ, it implies that $\phi_{J_0}^{-1}(a) \in J_0$ and 0 share more than one direct descendent in J_0. However, this would [11] imply $\tilde{e}_0 \cdot \tilde{e}_{\phi_{J_0}^{-1}(a)} \leq -1 < 0$, violating the non-negativity of the intersection numbers between distinct type I exceptional classes.

By the inductive assumption on the auxiliary statement $\diamondsuit$, we know that $\mathbf{Y}(\Gamma')$ is a smooth fibration over $\mathbf{Y}(\Gamma'_{J_0})$ under $\check{\pi}_{J_0}$.

Because the chain-like nature of Γ'_{J_0}, $\mathbf{Y}(\Gamma'_{J_0})\times_{\big((\mathcal{X}/B)_2/(\mathcal{X}/B)_1\big)_{|J_0|}} \big(\times_{(\mathcal{X}/B)_1}^{|J_0|} D_0\big)$ is the $codim_{\mathbf{C}}\Gamma'_{J_0}$ partial diagonal of the $(\mathbf{P}^1)^{|J_0|}$ bundle $\times_{(\mathcal{X}/B)_1}^{|J_0|} D_0$ (i.e. demanding that the $\mathbf{P}^1$ coordinates marked by indexes within the same connected component of the chain Γ'_{J_0} to be equal). And therefore it is a regular intersection (of codimension $codim_{\mathbf{C}}\Gamma_0$) in $\times_{(\mathcal{X}/B)_1}^{|J_0|} D_0$ and is of codimension $|J_0| = codim_{\mathbf{C}}\Gamma'_{J_0}$ in $\mathbf{Y}(\Gamma'_{J_0})$.

Then by the fact that $\mathbf{Y}(\tilde{\Gamma}_{\tilde{e}_0})$ is the inverse image of $\times_{(\mathcal{X}/B)_1}^{|J_0|} D_0$ under $\check{\pi}_{J_0}$, the fiber product (also the pull-back of the smooth fibration $\mathbf{Y}(\Gamma') \mapsto \mathbf{Y}(\Gamma'_{J_0})$

$$\mathbf{Y}(\Gamma') \times_{\big((\mathcal{X}/B)_2/(\mathcal{X}/B)_1\big)_{|J_0|}} \big(\times_{(\mathcal{X}/B)_1}^{|J_0|} D_0\big) = \mathbf{Y}(\Gamma') \cap \mathbf{Y}(\tilde{\Gamma}_{\tilde{e}_0}) = \cap_{0\leq i\leq n}\mathbf{Y}(\tilde{\Gamma}_{\tilde{e}_i})$$

is irreducible, smooth of codimension $codim_{\mathbf{C}}\mathbf{Y}(\Gamma') + |J_0| = codim_{\mathbf{C}}\mathbf{Y}(\Gamma') + codim_{\mathbf{C}}\tilde{\Gamma}_{\tilde{e}_0} = codim_{\mathbf{C}}\mathbf{Y}(\tilde{\Gamma}) = codim_{\mathbf{C}}\tilde{\Gamma}$ in $(\mathcal{X}/B)_{n+1} = \big((\mathcal{X}/B)_2/(\mathcal{X}/B)_1\big)_n$.

Correspondingly by a similar inductive argument the locally closed $\mathbf{Y}_{\tilde{\Gamma}}$ is equal to $\mathbf{Y}_{\Gamma'} \cap \mathbf{Y}_{\tilde{\Gamma}_{e_0}} = \cap_{0\leq i\leq n}\mathbf{Y}_{\tilde{\Gamma}_{\tilde{e}_i}}$, a Zariski dense subset of $\mathbf{Y}(\tilde{\Gamma})$ and therefore a locally closed subset of $(\mathcal{X}/B)_{n+1}$.

By adding 1 back to all the indexes, we find that $\mathbf{Y}(\Gamma) = \mathbf{Y}(\tilde{\Gamma})$ is an irreducible regular intersection $\cap_{1\leq i\leq n+1}\mathbf{Y}(\Gamma_{e_i})$ of codimension $codim_{\mathbf{C}}\Gamma$ in

[10] Refer to fig.4 on page 47 for an example.

[11] By a similar calculation as was performed in the proof of proposition 3.

$(\mathcal{X}/B)_{n+1}$ and $\mathbf{Y}_\Gamma = \cap_{1\leq i\leq n+1}\mathbf{Y}_{\Gamma_{e_i}}$ is an open subset of $\mathbf{Y}(\Gamma)$. So we have proved the proposition for $n+1$.

Now let us prove the auxiliary statement $\diamondsuit$ on the smoothness of the restricted morphism $\breve{\pi}_K$. Let K be an index subset of $\{1, 2, \cdots, n+1\}$. We show that $\breve{\pi}_K : \mathbf{Y}(\Gamma) \mapsto (\mathcal{X}/B)_{|K|}$ maps smoothly onto $\mathbf{Y}(\Gamma_K)$, for the $\Gamma_K \in adm(|K|)$ constructed earlier.

Firstly, if $K = \{1, \cdots, n+1\}$ itself, then $\Gamma_K = \Gamma$ itself and the map is an isomorphism. From now on we may assume $|K| \leq n$. Denote $I = \{1, 2, \cdots, n\}$. By induction hypothesis $\mathbf{Y}(\Gamma_I)$ maps onto $\mathbf{Y}(\Gamma_{K\cap I})$ smoothly under $\breve{\pi}_{K\cap I}$ and we have the following commutative diagram,

$$\begin{array}{ccccccc}
\mathbf{Y}(\Gamma) & \subset & (\mathcal{X}/B)_{n+1} & \xrightarrow{\breve{\pi}_K} & (\mathcal{X}/B)_{|K|} & \supset & \mathbf{Y}(\Gamma_K) \\
 & & \Big\downarrow{\scriptstyle \breve{\pi}_I} & & \Big\downarrow & & \Big\downarrow \\
\mathbf{Y}(\Gamma_I) & \subset & (\mathcal{X}/B)_n & \xrightarrow{\breve{\pi}_{I\cap K}} & (\mathcal{X}/B)_{|K\cap I|} & \supset & \mathbf{Y}(\Gamma_{K\cap I})
\end{array}$$

It is easy to see that $\Gamma_I \in adm(n)$ can be viewed as the admissible sub-graph [12] formed by restricting to the first n vertexes (and the one-edges between them) of Γ. Then by remark 1 the forgetful map (i.e. forgetting the last index $n+1$) $\breve{\pi}_I : \mathbf{Y}(\Gamma) \mapsto \mathbf{Y}(\Gamma_I)$ is smooth of relative dimension zero (i.e. isomorphic), dimension one (a $\mathbf{P}^1$ bundle), or dimension two[13], depending on whether $n+1$ has two, one or no direct ascendent(s) in I. Thus the map is smooth.

The smoothness of $\breve{\pi}_K : \mathbf{Y}(\Gamma) \mapsto \mathbf{Y}(\Gamma_K)$ follows from the smoothness of both $\mathbf{Y}(\Gamma_I) \mapsto \mathbf{Y}(\Gamma_{K\cap I})$ (by the induction hypothesis) and $\mathbf{Y}(\Gamma) \mapsto \mathbf{Y}(\Gamma_I)$, and the commutativity of the above diagram. The surjectivity of the map $\breve{\pi}_K : \mathbf{Y}(\Gamma) \mapsto \mathbf{Y}(\Gamma_K)$ follows from the fact [14] that all the fiberwise smooth and irreducible type I exceptional curves over Y_Γ are mapped to smooth and irreducible type I curves (or points) dual to $e_i^K, 1 \leq i \leq |K|$ under $(\mathcal{X}/B)_{n+1}/(\mathcal{X}/B)_n \mapsto (\mathcal{X}/B)_{|K|+1}/(\mathcal{X}/B)_{|K|}$. By the induction hypothesis of the proposition, $\mathbf{Y}_{\Gamma_K}$ has been the locus over which the type I curves representing $e_i^K, 1 \leq i \leq |K|$ co-exist as smooth curves. As $\mathbf{Y}_\Gamma \mapsto \mathbf{Y}_{\Gamma_K}$ is onto, the closure $\mathbf{Y}(\Gamma)$ has to be mapped onto the closure $\overline{\mathbf{Y}(\Gamma_K)} = \mathbf{Y}(\Gamma_K)$. So the inductive proof of the auxiliary statement $\diamondsuit$ has been complete. $\square$

Remark 3 *If we desire to minimize the dependence to the reference [Liu1], one may take an alternative route. One may turn lemma 3 and proposition 4 into constructions/definitions and use them to define the relative admissible strata* $\mathbf{Y}(\Gamma_{e_i})$, $\mathbf{Y}_{\Gamma_{e_i}}$, $\mathbf{Y}(\Gamma)$ *and* $\mathbf{Y}_\Gamma$, *etc. Then one may deduce all the basic properties from them. At the end we may take* $B = pt$ *to recover the usual* $Y(\Gamma)$ *and* Y_Γ *as a special case.*

[12] It was denoted by $\Gamma(-1)$ in [Liu1].

[13] If necessary, please consult the inductive construction of $Y(\Gamma)$ on page 418-419 of [Liu1] for more details. in that construction, a dependence of the fiber bundle structure upon the number of direct ascendents of $n+1$ was discussed in more details.

[14] which can be checked directly.

Remark 4 *We have remarked that the locus $Y(\Gamma_{e_i}) \subset M_n$ is the smooth locus over which the type I exceptional class becomes effective. So $Y(\Gamma) = \cap_{1\leq i\leq n} Y(\Gamma_{e_i})$ is the locus over which all the $e_1, e_2, \cdots, e_n$ become effective. Likewise, $Y_\Gamma \subset Y(\Gamma) \subset M_n$ is the locally closed locus over which the classes $e_1, e_2, \cdots, e_n$ co-exist as smooth and irreducible type I exceptional curves.*

Remark 5 *In the intersection $\cap_{1\leq i\leq n} Y(\Gamma_{e_i})$, we may ignore all the i such that $e_i^2 = -1$. Because each of such $e_i = E_i$ is a -1 class, the corresponding i has no direct descendent in Γ_{e_i}. We have $\Gamma_{e_i} = \gamma_n$ and $Y(\Gamma_{e_i}) = Y(\gamma_n) = M_n$. So the intersection with these $Y(\Gamma_{e_i})$ can be skipped.*

From now on we will make use of this simple observation implicitly. In the latter sections, we will use proposition 4 frequently and view $Y(\Gamma)$ as the locus of co-existence of all the type I exceptional classes e_i, over which they all become effective.

3 The Construction of the Quotient Bundle Based on the $\mathbf{P}^1$ Fibrations of Universal Curves

In this section, let $\Gamma \in adm(n)$ be an n-vertex admissible graph and let $Y(\Gamma)$ be the closure of the admissible stratum associated to Γ, as was described in section 2. Let $M_{n+1} \times_{M_n} Y(\Gamma) \mapsto Y(\Gamma)$ be the fiber bundle of algebraic surfaces over $Y(\Gamma)$ induced by $M_{n+1} \mapsto M_n$ through the pull-back map of $Y(\Gamma) \subset M_n$. Let $e_1, e_2, \cdots, e_n$ denote the type I exceptional classes associated to $Y(\Gamma)$. As usual, we let $e_{k_1}, e_{k_2}, e_{k_3}, \cdot, e_{k_i}, \cdots, e_{k_p}$, $k_1 < k_2 < \cdots k_p$, $1 \leq i \leq p$, denote the type I exceptional classes which pair negatively with $C - \mathbf{M}(E)E$. Because each e_i is effective and is represented by a unique curve over each point of $Y(\Gamma_{e_i})$, the notation $\Xi_i \mapsto Y(\Gamma_{e_i})$ has been used in [Liu1], [Liu3], [Liu5] to denote the $\mathbf{P}^1$ fibration (embedded in the fiber bundle $M_{n+1} \times_{M_n} Y(\Gamma_{e_i}) \mapsto Y(\Gamma_{e_i})$) representing the universal curves of e_i.

In section 5, proposition 9 of [Liu5], we had analyzed the canonical algebraic family Kuranishi models of two classes $C - \mathbf{M}(E)E - \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i$ and $C - \mathbf{M}(E)E$, $(\Phi_{\mathcal{V}^\circ_{canon}\mathcal{W}^\circ_{canon}}, \mathcal{V}^\circ_{canon}, \mathcal{W}^\circ_{canon})$ and $(\Phi_{\mathcal{V}_{canon}\mathcal{W}_{canon}}, \mathcal{V}_{canon}, \mathcal{W}_{canon})$ under the assumption[15] that $\mathcal{R}^1\pi_*\big(\mathcal{E}_C\big) = \mathcal{R}^2\pi_*\big(\mathcal{E}_C\big) = 0$.

We know $\mathcal{V}^\circ_{canon} = \mathcal{V}_{canon}$ but $\mathcal{W}^\circ_{canon}$ and $\mathcal{W}_{canon}$ differ from each other. In fact we have the four term exact sequence [16],

$$0 \mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}} \otimes \mathcal{E}_{C-\mathbf{M}(E)E}\big) \mapsto \mathcal{W}^\circ_{canon} \mapsto \mathcal{W}_{canon} \mapsto \mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}} \otimes \mathcal{E}_{C-\mathbf{M}(E)E}\big) \mapsto 0.$$

[15]We will assume that these conditions hold for the class C throughout the paper.
[16]by proposition 9 of [Liu5].

In particular their difference in the K group of $\mathbf{P}(\mathbf{V}_{canon})\times_{M_n} Y(\Gamma)$ can be represented by $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes \mathcal{E}_{C-\mathbf{M}(E)E}\big)-\mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum_{i\leq k}\Xi_{k_i}}\otimes \mathcal{E}_{C-\mathbf{M}(E)E}\big)$.

Because the canonical sections s°_{canon} and s_{canon} and the $\mathbf{H}-$twisted bundle map $\pi_X^*\mathbf{W}^{\circ}_{canon}\otimes\mathbf{H}|_{X\times_{M_n}Y(\Gamma)}\mapsto \pi_X^*\mathbf{W}_{canon}\otimes\mathbf{H}|_{X\times_{M_n}Y(\Gamma)}$ over $X\times_{M_n}Y(\Gamma)=\mathbf{P}(\mathbf{V}_{canon})\times_{M_n}Y(\Gamma)$ play important roles in the paper, it is vital for us to study the map $\mathbf{W}^{\circ}_{canon}\mapsto\mathbf{W}_{canon}$ in more details.

The $\mathbf{P}^1$ fibration $\Xi_i\mapsto Y(\Gamma_{e_i})$ may have singular fibers which are trees of $\mathbf{P}^1$ curves. Despite that the invertible sheaf $\mathcal{O}_{\Xi_{k_i}}(-\mathbf{M}(E)E)$ is of negative relative degree along $\Xi_{k_i}\mapsto Y(\Gamma_{e_{k_i}})$, the kernel vector spaces $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}(-\mathbf{M}(E)E)\otimes \mathcal{E}_C\otimes k(y)\big)$ may not always be the zero vector spaces and the canonical bundle map $\mathbf{W}^{\circ}_{canon}\mapsto\mathbf{W}_{canon}$ may fail to be injective over some sub-locus of $Y(\Gamma)$.

It is the goal of this section to construct an algebraic quotient bundle $\mathbf{V}_{quot}\mapsto \mathbf{P}(\mathbf{V}_{canon})\times_{M_n}Y(\Gamma)$ of $\mathbf{W}_{canon}|_{Y(\Gamma)\times T(M)}$ of rank[17] $-p+\sum_{1\leq i\leq p}e_{k_i}\cdot(-\mathbf{M}(E)E-\sum_{1\leq j<i\leq p}e_{k_j})$ and identify its total Chern class explicitly.

To construct $\mathbf{V}_{quot}$, we consider the torsion free part of the coherent sheaf $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}(-\mathbf{M}(E)E)\otimes\mathcal{E}_C\big)$ and show that,

Claim: The torsion free part of the first right derived image sheaf $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}(-\mathbf{M}(E)E)\otimes \mathcal{E}_C\big)$ is locally free.

The proof of the claim will appear in section 3.2 proposition 5.

Once we know that the torsion free sheaf is locally free, we denote it by $\mathcal{V}_{quot}$ and the corresponding vector bundle is our desired $\mathbf{V}_{quot}$.

The key idea for the explicit determination of its Chern classes is to consider the relative minimal model $\tilde{\Xi}_i$ of Ξ_i (see proposition 5.1 on page 442 of [Liu1]), which has a structure of $\mathbf{P}^1$ bundle over $Y(\Gamma_{e_i})$. The $\mathbf{P}^1$ fibration $\Xi_i\mapsto Y(\Gamma_{e_i})$ can be viewed as some consecutive blowing up from $\tilde{\Xi}_i$ along some codimension two sub-loci [18] determined by the graph Γ.

From the brief discussion in subsection 3.0.1 below on the torsion free quotient, we know that there is a canonical surjection,

$$\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}(-\mathbf{M}(E)E)\otimes\mathcal{E}_C\big)\mapsto(\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}(-\mathbf{M}(E)E)\otimes\mathcal{E}_C\big))_{torfree}=\mathcal{V}_{quot}.$$

By composing with the surjective sheaf morphism $\mathcal{W}_{canon}|_{Y(\Gamma)\times T(M)}\mapsto \mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}(-\mathbf{M}(E)E)\otimes\mathcal{E}_C\big)$, we get the surjection $\mathcal{W}_{canon}|_{Y(\Gamma)\times T(M)}\mapsto \mathcal{V}_{quot}$. Because both sheaves are locally free[19], we have the vector bundle short exact sequence over $Y(\Gamma)\times T(M)$, $0\mapsto\underline{\mathbf{W}}_{canon}\mapsto\mathbf{W}_{canon}|_{Y(\Gamma)\times T(M)}\mapsto \mathbf{V}_{quot}\mapsto 0$, where $\underline{\mathbf{W}}_{canon}$ is defined to be the kernel bundle of $\mathbf{W}_{canon}|_{Y(\Gamma)\times T(M)}\mapsto \mathbf{V}_{quot}$.

Definition 2 *Define* $\underline{\mathbf{W}}_{canon}$ *to be the kernel bundle of* $\mathbf{W}_{canon}|_{Y(\Gamma)\times T(M)}\mapsto \mathbf{V}_{quot}\mapsto 0$.

[17]By applying curve Riemann-Roch to the fibers of $\sum_{1\leq i\leq p}\Xi_{k_i}$.

[18]Even though it is possible to determine the sub-loci, the explicit form of this loci is not crucial to us.

[19]by the claim above.

Remark 6 *This short exact sequence will plays an essential role in our proof of the main theorem in section 6.*

Because the composition $\mathbf{W}^{\circ}_{canon}|_{Y(\Gamma)\times T(M)} \mapsto \mathbf{W}_{canon}|_{Y(\Gamma)\times T(M)} \mapsto \mathbf{V}_{quot}$ is the zero map, $\mathbf{W}^{\circ}_{canon}|_{Y(\Gamma)\times T(M)} \mapsto \mathbf{W}_{canon}|_{Y(\Gamma)\times T(M)}$ factors through the kernel $\underline{\mathbf{W}}_{canon}$ and,

Lemma 4 *The induced bundle map $\mathbf{W}^{\circ}_{canon}|_{Y(\Gamma)\times T(M)} \mapsto \underline{\mathbf{W}}_{canon}$ is injective over a Zariski open subset $U = Y_\Gamma \times T(M)$ of $Y(\Gamma)\times T(M)$.*

Proof of lemma 4: The kernel spaces $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}} \otimes \mathcal{E}_{C-\mathbf{M}(E)E} \otimes k(y)\big)$, $y \in Y(\Gamma)\times T(M)$, of $\mathbf{W}^{\circ}_{canon}|_{Y(\Gamma)\times T(M)} \mapsto \mathbf{W}_{canon}|_{Y(\Gamma)\times T(M)}$ (see proposition 9 of [Liu5]) is supported "away" from the open dense subset $Y_\Gamma \times T(M) \subset Y(\Gamma)\times T(M)$ over which the fibrations $\Xi_{k_i} \mapsto Y(\Gamma)$ are smooth and irreducible for all $1 \leq i \leq p$. Since the fibers of the restricted fibrations $\Xi_{k_i} \times_{Y(\Gamma_{e_{k_i}})} Y_\Gamma \mapsto Y_\Gamma$, $1 \leq i \leq p$, remain smooth and irreducible throughout $Y_\Gamma \times T(M)$, the vanishing of $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}} \otimes \mathcal{E}_{C-\mathbf{M}(E)E}\big)|_{Y_\Gamma\times T(M)}$ is due to the negative relative degrees of the invertible $\mathcal{E}_{C-\mathbf{M}(E)E}$ on all the Ξ_{k_i}, i.e. $e_{k_i}\cdot(C-\mathbf{M}(E)E) < 0$. So $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}} \otimes \mathcal{E}_{C-\mathbf{M}(E)E}\big)$ is a torsion sheaf [20] over $Y(\Gamma)\times T(M)$.

On the other hand, the sheaf map $\mathcal{W}^{\circ}_{canon}|_{Y(\Gamma)\times T(M)} \mapsto \mathcal{W}_{canon}|_{Y(\Gamma)\times T(M)}$ factors through the intermediate $\underline{\mathcal{W}}_{canon}$. So the sheaf morphism $\mathcal{W}^{\circ}_{canon} \mapsto \underline{\mathcal{W}}_{canon}$ is injective over $Y_\Gamma \times T(M)$. $\square$

3.0.1 A Short Remark about the Torsion Free Sub-sheaves

Let $\mathcal{F}$ be a coherent sheaf over a smooth and connected scheme Y. Let $U = Spec(R)$ be an affine open subspace of Y. Then R is an integral domain and let K denote the field of fractions. Over U the coherent sheaf $\mathcal{F}$ is the sheaf associated to a finite R-module N. Recall that (e.g [Fr]) the generic rank of $\mathcal{F}$ is defined to be the rank of $N \otimes_R K$.

Define $\mathcal{F}^* = \mathcal{HOM}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y)$ to be the dual sheaf. Then there is a natural morphism $\mathcal{F} \mapsto \mathcal{F}^{**}$. Define $\mathcal{F}_{torfree}$ to be the image of $\mathcal{F}$ in $\mathcal{F}^{**}$, and it is a torsion free sheaf.

On the other hand by corollary 21 on page 44 of [Fr] we have the following short exact sequence,

$$0 \mapsto \mathcal{F}_{tor} \mapsto \mathcal{F} \mapsto \mathcal{F}_{torfree} \mapsto 0,$$

where the cokernel $\mathcal{F}_{tor}$ is the torsion sub-sheaf of $\mathcal{F}$ over Y. From now on, we call $\mathcal{F}_{torfree}$ the torsion free quotient (part) of $\mathcal{F}$.

In general the inclusion $\mathcal{F}_{torfree} \subset \mathcal{F}^{**}$ is not always an equality. A torsion free sheaf $\mathcal{F} \cong \mathcal{F}^{**}$ under the injection is called a reflexive sheaf. It is well known that locally free sheaves are reflexive.

[20] As it is a torsion sub-sheaf of a locally free sheaf, it vanishes. But the sheaf injection $\mathcal{W}^{\circ}_{canon} \mapsto \mathcal{W}_{canon}$ does not induce a bundle injection, because generally speaking $\otimes k(y)$ is not left-exact.

Lemma 5 *Let Y be a reduced, smooth and connected scheme. Let $\mathcal{F}$ be coherent and let $\mathcal{E}$ be locally free such that generic rank of $\mathcal{F}$=rank of $\mathcal{E}$. Let $\mathcal{F} \mapsto \mathcal{E} \mapsto 0$ be a sheaf surjection, then $\mathcal{F}_{torfree} \cong \mathcal{E}$.*

Proof of lemma 5: Consider the following commutative diagram,

$$\begin{array}{ccccc} \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ \downarrow & & \downarrow \cong & & \\ \mathcal{F}^{**} & \longrightarrow & \mathcal{E}^{**} & & \end{array}$$

The surjectivity of $\mathcal{F} \mapsto \mathcal{E} \cong \mathcal{E}^{**}$ implies that the image of $\mathcal{F}$ in $\mathcal{F}^{**}$, $\mathcal{F}_{torfree}$, maps surjectively onto $\mathcal{E}^{**}$. Because $\mathcal{F}_{torfree}$ and $\mathcal{E}^{**}$ are of the same generic rank, the kernel of $\mathcal{F}_{torfree} \mapsto \mathcal{E}^{**} \mapsto 0$ has to be a torsion sheaf. But a torsion sheaf can never maps injectively into a torsion free sheaf $\mathcal{F}_{torfree}$. So it must vanish and $\mathcal{F}_{torfree} \cong \mathcal{E}^{**}$ and therefore $\mathcal{F}_{torfree}$ is isomorphic to $\mathcal{E}$ itself. $\square$

In other words, $\mathcal{E}$ is isomorphic to the torsion free quotient of $\mathcal{F}$.

Lemma 6 *Let Y be a smooth, connected and reduced scheme. Let $\mathcal{F}$ be coherent and let $\mathcal{E}$ be locally free and generic rank of $\mathcal{F}$=rank of $\mathcal{E}$. Let $0 \mapsto \mathcal{E} \mapsto \mathcal{F}$ be a sheaf injection such that $\bullet \otimes k(y)$ is left exact for all the closed points $y \in Y$. Then $\mathcal{E} \cong \mathcal{F}_{torfree}$ under the composition $\mathcal{E} \mapsto \mathcal{F} \mapsto \mathcal{F}_{torfree}$.*

In such a case, $\mathcal{E} \cong \mathcal{F}_{torfree}$ induces a morphism $\mathcal{F}_{torfree} \mapsto \mathcal{F}$ through $\mathcal{E}$ and we may write $\mathcal{F} = \mathcal{F}_{torfree} \oplus \mathcal{F}_{tor}$ and call $\mathcal{F}_{torfree}$ and $\mathcal{F}_{tor}$ the torsion free and the torsion summands of $\mathcal{F}$.

Proof: Consider the following commutative diagram,

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{F} \\ \downarrow \cong & & \downarrow \\ \mathcal{E}^{**} & \longrightarrow & \mathcal{F}^{**} \end{array}$$

To show that $\mathcal{E} \cong \mathcal{F}_{torfree}$, it suffices to show that $\mathcal{E} \cong \mathcal{E}^{**} \cong \mathcal{F}^{**}$.

Because $\mathcal{E}$ and $\mathcal{F}$ are of the same generic rank, the cokernel of the short exact sequence $0 \mapsto \mathcal{E} \mapsto \mathcal{F} \mapsto \mathcal{R} \mapsto 0$ is torsion. Applying the contravariant left exact functor $\mathcal{HOM}_{\mathcal{O}_Y}(\bullet, \mathcal{O}_Y)$ to this short exact sequence and observing $\mathcal{HOM}_{\mathcal{O}_Y}(\mathcal{R}, \mathcal{O}_Y) = 0$ because of its torsion nature, we have

$$0 \mapsto \mathcal{F}^* \mapsto \mathcal{E}^* \mapsto \mathcal{R}' \mapsto 0.$$

The cokernel $\mathcal{R}'$ appears as the $\mathcal{HOM}_{\mathcal{O}_Y}(\bullet, \mathcal{O}_Y)$ functor is not always right exact. By proposition 24 on page 45 of [Fr], $\mathcal{F}^*$ is reflexive and torsion free. On the other hand the $k(y)$ vector space morphism $\mathcal{F}^* \otimes k(y) \mapsto \mathcal{E}^* \otimes k(y)$ is equivalent to $HOM_{k(y)}(\mathcal{F} \otimes k(y), k(y)) \mapsto HOM_{k(y)}(\mathcal{E} \otimes k(y), k(y))$ and is always surjective since $\mathcal{E} \otimes k(y) \mapsto \mathcal{F} \otimes k(y)$ is always injective for all the closed points $y \in Y$. As $\bullet \otimes k(y)$ is right exact, the $rank_{k(y)} \mathcal{R}' \otimes k(y) = 0$ for all the closed points y of Y. Then by exercise II.5.8 on page 125 of [Ha] and the fact that Y is reduced, $\mathcal{R}'$ is locally free of rank 0, so $\mathcal{R}'$ vanishes.

Therefore $\mathcal{F}^* \cong \mathcal{E}^*$. By dualizing this equality again we get the desired $\mathcal{E}^{**} \cong \mathcal{F}^{**}$. The lemma is proved. $\Box$

The reader may consult page 42-46 of [Fr] for some basic knowledge about torsion free sheaves.

In our paper we consider the torsion free quotient of coherent sheaves which are the derived image sheaves of invertible sheaves along a union of $\mathbf{P}^1$ fibrations or along finite morphisms. In the proof of proposition 5, we show that these torsion free sheaves are in fact locally free. When this situation occurs, we denote $\mathcal{F}_{torfree}$ by an alternative notation $\mathcal{F}_{free}$ (or $(\mathcal{F})_{free}$) to indicate that it is not only torsion free, but is actually locally free.

3.1 The Construction of $\mathbf{P}^1$ fiber bundles $\tilde{\Xi}_i \mapsto Y(\Gamma_{e_i})$

Let $\Gamma \in adm(n)$ be an n-vertex admissible graph and let $Y(\Gamma) \subset M_n$ be the smooth closure of the corresponding admissible stratum Y_Γ. As usual let $m_1, m_2, \cdots, m_n$ be the multiplicities satisfying $0 < m_a \leq m_b$ whenever $1 \leq a \leq b \leq n$. We assume that such a multiple covered $\mathbf{M}(E)E = \sum_{1\leq i\leq n} m_i E_i$ has been fixed.

Let $\pi : \Xi_i \mapsto Y(\Gamma_{e_i})$ be [21] the $\mathbf{P}^1$ fibration in $M_{n+1} \times_{M_n} Y(\Gamma_{e_i})$ representing the universal exceptional curves dual to e_i. In the following we would like to construct smooth $\mathbf{P}^1$ fiber bundles $\tilde{\pi} : \tilde{\Xi}_i \mapsto Y(\Gamma_{e_i})$ birational to Ξ_i for $1 \leq i \leq n$. To simplify our notations, we would like to drop the restriction symbol[22] and denote their restrictions to the sub-locus $Y(\Gamma) = \cap_{1\leq i\leq p} Y(\Gamma_{e_i}) \subset Y(\Gamma_{e_i})$, $\Xi_i|_{Y(\Gamma)}$ or $\tilde{\Xi}_i|_{Y(\Gamma)}$ by the same notations.

Given a subscript $1 \leq i \leq n$, we define I_i in the following,

Definition 3 *Define the index set I_i to be the set of all the subscripts of E appearing in $e_i = E_i - \sum_{j_i} E_{j_i}$. I.e. the union of $\{i\}$ and all the direct descendent indexes of i in Γ.*

Given an index subset $I \subset \{1, 2, \cdots, n\}$, by corollary 1 in section 2 there exists the canonical lifting $\tilde{\pi}_I : M_n \mapsto M_{|I|}$ of $\pi_I : M_n \mapsto M^{|I|}$ and it induces the canonical map $Y(\Gamma) \mapsto M_{|I|}$ by composing $Y(\Gamma) \hookrightarrow M_n$ and $M_n \mapsto M_{|I|}$.

By taking $I = \{1, 2, \cdots, i-1\} \cup I_i$ in the above setting, we may construct the total space of the $\mathbf{P}^1$ fibration $\tilde{\Xi}_i$ as a divisor in the fiber product $M_{i+|I_i|} \times_{M_{i-1+|I_i|}} Y(\Gamma_{e_i})$.

Lemma 7 *There exists an $\mathbf{P}^1$ fibration over $Y(\Gamma_{e_i})$, $\tilde{\pi} : \tilde{\Xi}_i \mapsto Y(\Gamma_{e_i})$ such that*

(i). $\tilde{\Xi}_i$ is pulled back from a $\mathbf{P}^1$ sub-fibration of $M_{i+|I_i|} \mapsto M_{i-1+|I_i|}$ by $Y(\Gamma_{e_i}) \stackrel{\tilde{\pi}_I}{\mapsto} M_{|I|} = M_{i-1+|I_i|}$.

(ii). $\tilde{\Xi}_i \mapsto Y(\Gamma_{e_i})$ has a structure of $\mathbf{P}^1$ fiber bundle over $Y(\Gamma_{e_i})$.

[21] The construction of Ξ_i will be outlined in the proof of lemma 7.

[22] The reader should be able to recover the restriction notation from the base space we are using.

(iii). The space Ξ_i maps birationally onto $\tilde{\Xi}_i$ and the birational map $\Xi_i \mapsto \tilde{\Xi}_i$ is a consecutive blowing ups along codimension two smooth centers.

Thus $\tilde{\Xi}_i$ is the relative minimal model of Ξ_i.

Proof of lemma 7: Because $I = \{1, 2, \cdots, i-1\} \cup I_i$, $|I| = i - 1 + |I_i|$. Consider an ordering preserving bijection $\phi : I \mapsto \{1, 2, \cdots, |I|\}$. Then $\phi(j) = j$ for $j \leq i$. Consider an $|I|-$vertex fan-like admissible graph $\Gamma_i \in adm(|I|)$ with one-edges from $i-$th vertex to all the vertexes marked in $\phi(I_i - \{i\})$. Then the $i-$th vertex is the direct ascendent of all the other vertexes in Γ_i marked by $\phi(I_i - \{i\})$ and it is the only direct ascendent vertex in Γ_i. So $Y(\Gamma_{e_i})$ is mapped onto $Y(\Gamma_i)$ under $Y(\Gamma_{e_i}) \mapsto M_{|I|}$. This can be seen by the construction of $\mathbf{Y}(\Gamma_{e_i})$ in lemma 3 of section 2 as the pre-image [23] of $\mathbf{Y}(\Gamma_i) \cong \times^{|I_i|-1}_{M_i/M_{i-1}} D_i$ under[24] $\breve{\pi}_I$. Under the surjection $Y(\Gamma_{e_i}) \mapsto Y(\Gamma_i)$, the open subset $Y_{\Gamma_{e_i}}$ is mapped surjectively onto Y_{Γ_i}.

As a subspace of the space $M_{|I|}$, $Y(\Gamma_i)$ is characterized as the existence locus of the type I exceptional class $E_i - \sum_{|I_i| \geq j \geq 2} E_{j+i-1}$. So there exists a $\mathbf{P}^1$ fibration of universal curves over $Y(\Gamma_i) \subset M_{|I|}$, $\mathcal{C}_i \mapsto Y(\Gamma_i)$, whose fiber over $b \in Y(\Gamma_i) \subset M_{|I|}$ is the type I exceptional curve representing $E_i - \sum_{|I_i| \geq j \geq 2} E_{j+i-1}$ in the algebraic surface $M_{|I|+1}|_b$. So $\mathcal{C}_i \mapsto Y(\Gamma_i)$ can be viewed as a sub-fibration of $M_{|I|+1} \times_{M_{|I|}} Y(\Gamma_i)$. By pulling back $\mathcal{C}_i \mapsto Y(\Gamma_i)$ by $Y(\Gamma_{e_i}) \mapsto Y(\Gamma_i)$, we define $\tilde{\Xi}_i$ to be $\mathcal{C}_i \times_{Y(\Gamma_i)} Y(\Gamma_{e_i})$. Then the condition (i). holds by our construction.

To prove (ii)., it suffices to show [25] that $\mathcal{C}_i \mapsto Y(\Gamma_i)$ is a $\mathbf{P}^1$ fiber bundle. By a special case of lemma 1, we have $M_{|I|+1} \cong (M_{i+1}/M_i)_{|I|-i+1} = (M_{i+1}/M_i)_{|I_i|}$. On the other hand, the exceptional divisor $D_i \subset M_{i+1}/M_i$ blown up from the relative diagonal $M_i \mapsto M_i \times_{M_{i-1}} M_i$ has a $\mathbf{P}^1$ bundle structure over M_i. So we have $(D_i/M_i)_{|I_i|} \subset (M_{i+1}/M_i)_{|I_i|}$. On the other hand, we have the commutative diagram,

$$\begin{array}{ccccc}
(D_i/M_i)_{|I_i|} & \longrightarrow & (M_{i+1}/M_i)_{|I_i|} & \cong & M_{|I_i|+i} \\
\big\downarrow & & \big\downarrow & & \\
(D_i/M_i)_{|I_i|-1} & \longrightarrow & (M_{i+1}/M_i)_{|I_i|-1} & \cong & M_{|I_i|+i-1}
\end{array}$$

Because $D_i \mapsto M_i$ is smooth of relative dimension one[26], the projection of fiber products $\times^{|I_i|}_{M_i} D_i \cong (D_i/M_i)_{|I_i|} \mapsto (D_i/M_i)_{|I_i|-1} \cong \times^{|I_i|-1}_{M_i} D_i \cong Y(\Gamma_i)$ has a $\mathbf{P}^1$ fiber bundle structure. It suffices to identify $(D_i/M_i)_{|I_i|}$ with $\mathcal{C}_i$.

By induction it is easy to see that the fiber bundle $M_{|I|+1} \mapsto M_{|I|}$ can be constructed from the trivial bundle $M_1 \times_{M_0} M_{|I|} = M \times M_{|I|} \mapsto M_{|I|}$ by $|I|-$

[23] Please refer to the proof of lemma 3 for more details.

[24] The space $D_i \mapsto M_i$ is the exceptional divisor by blowing up along $\Delta_{M_i} : M_i \subset M_i \times_{M_{i-1}} M_i$ and has a $\mathbf{P}^1$ bundle structure over M_i.

[25] An alternative way to achieve this is to check that the type I class $E_i - \sum_{2 \leq j \leq |I_i|} E_{j+i-1}$ can not be broken into two distinct type I classes.

[26] Here we are using the fact that the codimension one blowing ups are trivial.

consecutive blowing ups along the cross sections [27] of the intermediate fiber bundles $M_{k+1}\times_{M_k} M_{|I|}\mapsto M_{|I|}$, for $0\leq k\leq |I|-1$. Consider $f_{|I|-1,i}: M_{|I|}\mapsto M_i$ and the pulled-back fiber bundle $f^*_{|I|-1,i}D_i\subset M_{i+1}\times_{M_i} M_{|I|}$ is isomorphic to the exceptional divisor E_i of the $i-$th intermediate fiber bundle. Thus the projection $M_{|I|+1}/M_{|I|}\mapsto M_{i+1}\times_{M_i} M_{|I|}/M_{|I|}$ to the $i-$th intermediate space can be constructed by $|I|-i=|I_i|-1-$consecutive blowing ups along cross sections of the intermediate fiber bundles. When we restrict to the locus $Y(\Gamma_i)\subset M_{|I|}$, the $j-$th cross section, for all $1\leq j\leq |I_i|-1$, maps into the sub-bundle $f^*_{|I|-1,i}D_i\mapsto M_{|I|}$ and becomes a cross section of $f^*_{|I|-1,i}D_i|_{Y(\Gamma_i)}\mapsto Y(\Gamma_i)$.

On the other hand, by Chapter II corollary 7.15. of [Ha], the strict transform of the restriction of the exceptional divisor $E_i\times_{M_{|I|}} Y(\Gamma_i)\subset M_{i+1}\times_{M_i} Y(\Gamma_i)$ inside $M_{|I|+1}\times_{M_{|I|}} Y(\Gamma_i)$ is nothing but the $|I_i|-1-$consecutive blowing ups of $f^*_{|I|-1,i}D_i|_{Y(\Gamma_i)}$ along the $|I_i|-1$ distinct cross sections. Because the $\mathbf{P}^1$ fiber bundle is smooth of relative dimension one, all the blowing ups along cross sections are trivial. Thus its strict transform $\mathcal{C}_i$, representing $E_i-\sum_{|I_i|\geq j\geq 2} E_{j+i-1}$ in $M_{|I|+1}\times_{M_{|I|}} Y(\Gamma_i)$, is still isomorphic to $f^*_{|I|-1,i}D_i|_{Y(\Gamma_i)}$. The condition (ii). is proved.

In the following, we derive the conclusion (iii) based on a similar argument as above. Consider the projection map $f_{n-1,i}: M_n\mapsto M_i$ and the induced $\mathbf{P}^1$ bundle $f^*_{n-1,i}D_i\subset M_{i+1}\times_{M_i} M_n$ is the exceptional divisor E_i of the $i-$th intermediate fiber bundle in-between M_{n+1} and the trivial product $M\times M_n$. Similar to the above argument the map $M_{n+1}\mapsto M_{i+1}\times_{M_i} M_n$ can be constructed by $n-i-$consecutive blowing ups along cross sections of the $n-i-$intermediate fiber bundles. Similar to the above discussion to $\mathcal{C}_i$, Ξ_i is the strict transform of $E_i\times_{M_n} Y(\Gamma_{e_i})\subset M_{i+1}\times_{M_i} M_n$ under these consecutive blowing ups. Again by Chapter II corollary 7.15. of [Ha], Ξ_i can be identified with the $n-i-$ consecutive blowing ups of $F_0=f^*_{n-1,i}D_i\times_{M_n} Y(\Gamma_{e_i})$ along the intersections (of the intermediate blown up spaces from F_0) with the various cross sections in the intermediate fiber bundles [28] $M_{k+i+1}\times_{M_{k+i}} M_n\mapsto M_n$. Denote $C_0\subset F_0$ to be the first blowup center and inductively define $F_k=BlowUp_{C_{k-1}}F_{k-1}$. Denote $C_k\subset F_k$ to be the $k-$th blowup center, for k ranging in $0\leq k\leq n-i-1$. At the end we have $F_{n-i}=\Xi_i$ and it suffices to show that all the blowup centers C_k $(0\leq k\leq n-i-1)$ are smooth of codimension two/one in F_k.

Because C_k is the intersection of F_k with a cross section of the ambient fiber bundle $M_{k+i+1}\times_{M_{k+i}} M_n\mapsto M_n$, the projection $C_k\subset F_k\mapsto Y(\Gamma_{e_i})$ induces an isomorphism onto the image of the intersection locus. Suppose that C_k maps onto $Y(\Gamma_{e_i})$, then C_k must be a cross section of $F_k\mapsto Y(\Gamma_{e_i})$ and therefore [29] is smooth. This can only occur when $k+i+1$ is a direct descendent index of i throughout $Y(\Gamma_{e_i})$, which happens only when $k+i+1\in I_i$. If $k+i+1\not\in I_i$, then $k+i+1$ is not a direct descendent index of i in Γ_{e_i}. Consider a particular degeneration $\Gamma_{e_i;k+i+1}$ of Γ_{e_i} by adding a single one-edge

[27] They are pull-backs of the relative diagonals $\Delta_{M_{k+1}/M_k}: M_{k+1}\mapsto M_{k+1}\times_{M_k} M_{k+1}$ by $f_{|I|-1,k+1}: M_{|I|}\mapsto M_{k+1}$.

[28] Pulled-back from $M_{k+i+1}\mapsto M_{k+i}$ by $M_n\mapsto M_{k+i}$.

[29] By proposition 1 the space $Y(\Gamma_{e_i})$ is smooth.

from i to $k+i+1$. Then by proposition 1, $Y(\Gamma_{e_i;k+i+1}) \subset Y(\Gamma_{e_i})$ is a smooth divisor in $Y(\Gamma_{e_i})$. On the other hand, C_k is the intersection of F_k, i.e. the strict transform of F_0, with the cross section of $M_{k+i+1} \times_{M_{k+i}} M_n \mapsto M_n$ induced by the relative diagonal. So at the location where F_k intersects the cross section, the $k+i+1-$th blowing up in F_k determined by the intersection locus is located in the strict transform of the exceptional locus E_i of the $i-$th blowing up. By the interpretation of remark 1 this occurs exactly when $k+i+1$ becomes a (direct or indirect) descendent of i and so C_k maps onto $Y(\Gamma_{e_i;k+i+1})$. Therefore C_k is isomorphic to $Y(\Gamma_{e_i;k+i+1}) \subset Y(\Gamma_{e_i})$ and by proposition 1 it is smooth. When this occurs C_k is of codimension two in F_k. Then by induction $\Xi_i = F_{n-k}$ is an $n-k-$consecutive blowing up of F_0 along codimension two smooth centers.

Because $f_{|I|-1,i} \circ f_{n-1,|I|} = f_{n-1,i}$, we have $\tilde{\Xi}_i = (f_{n-1,|I|}|_{Y(\Gamma_{e_i})})^* \mathcal{C}_i = (f_{n-1,|I|}|_{Y(\Gamma_{e_i})})^* (f_{|I|-1,i})^* D_i = (f_{n-1,i}|_{Y(\Gamma_{e_i})})^* D_i = F_0$. So $\Xi_i = F_{n-k}$ projects onto $F_0 = \tilde{\Xi}_i$ and $\tilde{\Xi}_i$ is the relative minimal model of Ξ_i. This finishes the proof of (iii). $\square$

Remark 7 *Because some of these C_k are not cross sections and are not dominating $Y(\Gamma_{e_i})$, the blowing ups along those C_k cause the special fibers of $\Xi_i \mapsto Y(\Gamma_{e_i})$ to become a finite tree of normal-crossing $\mathbf{P}^1$.*

For a fixed i one may re-write the cohomology class $C - \mathbf{M}(E)E = C - \sum_{1 \le a \le n} m_a E_a$ as $C - \sum_{a \in I_i} m_a E_a - \sum_{a \notin I_i} m_a E_a$ and there is a canonical (up to rescaling of $\mathbf{C}^*$) sheaf morphism $\mathcal{E}_{C-\mathbf{M}(E)E} = \mathcal{E}_{C - \sum_{a \in I_i} m_a E_a - \sum_{a \notin I_i} m_a E_a} \mapsto \mathcal{E}_{C - \sum_{a \in I_i} m_a E_a - \sum_{a < i} m_a E_a}$ by tensoring with the defining sections of $\sum_{i < a \notin I_i} m_a E_a$ on $\mathcal{M}_{n+1}$.

The main reason that we introduce the fibrations $\tilde{\Xi}_{k_i}$, $1 \le i \le p$ for e_{k_i} is because of the following,

Lemma 8 *The sheaf $\mathcal{R}^0 \tilde{\pi}_* \big(\mathcal{O}_{\tilde{\Xi}_{k_i}} \otimes \mathcal{E}_{C - \sum_{a \in I_i} m_a E_a - \sum_{a < i} m_a E_a}\big)$ is the zero sheaf. The first derived image sheaf $\mathcal{R}^1 \tilde{\pi}_* \big(\mathcal{O}_{\tilde{\Xi}_{k_i}} \otimes \mathcal{E}_{C - \sum_{a \in I_i} m_a E_a - \sum_{a < i} m_a E_a}\big)$ is locally free.*

Proof: Based on the condition $e_{k_i} \cdot (C - \mathbf{M}(E)E) < 0$, the relative degree of the invertible sheaf $\mathcal{F} = \mathcal{E}_{C - \sum_{a \in I_i} m_a E_a - \sum_{a < i} m_a E_a}$ along $\tilde{\Xi}_{k_i} \mapsto Y(\Gamma_{e_{k_i}})$ is negative. Since by lemma 7, $\tilde{\Xi}_{k_i} \mapsto Y(\Gamma_{e_{k_i}})$ is a $\mathbf{P}^1$ fiber bundle. The negativity of the relative degree implies the vanishing of $\mathcal{R}^0 \tilde{\pi}_* \big(\mathcal{O}_{\tilde{\Xi}_{k_i}} \otimes \mathcal{E}_{C - \sum_{a \in I_i} m_a E_a - \sum_{a < i} m_a E_a}\big)$. On the other hand, the vanishing of the zero-th derived image sheaf implies $h^0(y, \mathcal{F}) = 0$ for $y \in Y(\Gamma_{e_{k_i}}) \times T(M)$ and by curve Riemann-Roch formula it implies the constancy of $h^1(y, \mathcal{F})$ throughout $Y(\Gamma_{e_{k_i}}) \times T(M)$. Then by chapter II, corollary 12.9 of [Ha] the sheaf $\mathcal{R}^1 \tilde{\pi}_* \big(\mathcal{E}_{C - \sum_{a \in I_i} m_a E_a - \sum_{a < i} m_a E_a}\big)$ is locally free. $\square$

The locally freeness of the derived image sheaves along the various $\tilde{\Xi}_{k_i} \mapsto Y(\Gamma_{e_{k_i}})$ enables us to find an explicit representative [30] of $[\mathcal{V}_{quot}] \in K_0(Y(\Gamma) \times T(M))$ in the following subsection.

3.2 The Locally Freeness of $\mathcal{V}_{quot}$ and its Explicit Representative in the K Group

In this subsection, we would like to prove the locally freeness of the torsion free quotient $\mathcal{V}_{quot}$ of $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}} \otimes \mathcal{E}_{C-\mathbf{M}(E)E}\big)$ over $Y(\Gamma) \times T(M)$ and we also give an explicit identification of $[\mathcal{V}_{quot}] \in K_0(Y(\Gamma) \times T(M))$.

Lemma 9 *The torsion free quotient of the coherent sheaf* $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\Xi_{k_p}} \otimes \mathcal{E}_{C-\sum_{a\leq n} m_aE_a}\big)$ *is locally free and is isomorphic to* [31] $\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}} \otimes \mathcal{E}_{C-\sum_{a\in I_{k_i}} m_aE_a}\big) \otimes \tilde{\pi}^*\mathcal{O}(-\sum_{a<k_i} m_aE_{a;k_i})$.

Proof of the lemma: The above sheaves are of the same generic rank, by a direct curve Riemann-Roch calculation along smooth fibers above $Y_\Gamma \times T(M)$

The locally freeness of $\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}} \otimes \mathcal{E}_{C-\sum_{a\in I_{k_i}} m_aE_a - \sum_{a<k_i} m_aE_a}\big)$ has been proved in lemma 8. The isomorphism

$$\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}} \otimes \mathcal{E}_{C-\sum_{a\in I_{k_i}} m_aE_a - \sum_{a<k_i} m_aE_a}\big) \cong \mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}} \otimes \mathcal{E}_{C-\sum_{a\in I_{k_i}} m_aE_a}\big) \otimes \tilde{\pi}^*\mathcal{O}(-\sum_{a\in k_i} m_aE_{a;k_i})$$

follows from the projection formula (exercise II.8.3. on page 253 of [Ha]) and the fact that $\mathcal{O}(-E_a)|_{\tilde{\Xi}_{k_i}} = \tilde{\pi}^*\mathcal{O}(E_{a;k_i})|_{Y(\Gamma)}$, for $a < k_i$.

To prove the lemma, by lemma 5 it suffices to prove that $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\Xi_{k_p}} \otimes \mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\big) \mapsto \mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}} \otimes \mathcal{E}_{C-\sum_{a\in I_{k_i}} m_aE_a - \sum_{a<k_i} m_aE_a}\big)$ is surjective.

Firstly by lemma 7 (iii). $\Xi_{k_i} \mapsto \tilde{\Xi}_{k_i}$ is a composite blowing down map, the push-forward of $\mathcal{O}_{\Xi_{k_i}}(-\sum_{n\geq a\geq k_i; a\notin I_{k_i}} m_aE_a)$ to $\tilde{\Xi}_{k_i}$ defines an ideal sheaf [32] of the sub-scheme $\mathcal{I}_{Z_t} \subset \mathcal{O}_{\tilde{\Xi}_{k_i}}$. To show the surjectivity of the original sheaf map, it suffices to show that $Z_t \mapsto Y(\Gamma_{e_{k_i}})$ is at most of relative dimension zero. I.e. the fibers of $Z_t \mapsto Y(\Gamma_{e_{k_i}})$ are either empty or are zero dimensional. In fact $\tilde{\Xi}_{k_i}$ is a $\mathbf{P}^1$ fiber bundle and all the fibers are smooth and irreducible. On the other hand Z_t supports over the image of $\sum_{n\geq a\geq k_i; a\notin I_{k_i}} m_aE_a$ in $\tilde{\Xi}_{k_i}$. So the only chance for $Z_t|_y$, $y \in Y(\Gamma)$ to be one dimensional is when $Z_t|_y$ supports over the whole $\tilde{\Xi}_{k_i}|_y$. This implies that the defining section of $\mathcal{O}(\sum_{n\geq a\geq k_i; a\notin I_{k_i}} m_aE_a)|_y$

[30] For the definition of $\mathcal{V}_{quot}$, please consult page 21.

[31] The symbol $E_{a;b}$, $a < b$, denote the exceptional divisor in M_n by blowing up the the $(a,b)-$th partial diagonal.

[32] The subscript t of the notation Z_t stands for "torsion" because Z_t is closely related to the torsion part of a sheaf.

is divisible by the defining section of $(E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}})|_y$. This implies the existence of an admissible graph $\Gamma', \Gamma' < \Gamma$ and $y \in Y_{\Gamma'}$, such that k_i is a descendent of some $a \geq k_i, a \notin I_{k_i}$. This is absurd because the axiom 2. of admissible graphs forbids k_i with $k_i < a$ to be the descendent of a.

Therefore $Z_t \mapsto Y(\Gamma)$ is generically empty and is at most relative dimension zero. Then by tensoring the defining exact sequence $0 \mapsto \mathcal{I}_{Z_t} \mapsto \mathcal{O}_{\tilde{\Xi}_{k_i}} \mapsto \mathcal{O}_{Z_t} \mapsto 0$ with $\mathcal{E}_{C-\sum_{a\in I_{k_i}} m_a E_a - \sum_{a<k_i} m_a E_a}$, we get the desired surjectivity from a portion of its derived long exact sequence,

$$\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{I}_{Z_t}\otimes\mathcal{E}_{C-\sum_{a\in I_{k_i}} m_a E_a - \sum_{a<k_i} m_a E_a}\big) \mapsto \mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}}\otimes\mathcal{E}_{C-\sum_{a\in I_{k_i}} m_a E_a - \sum_{a<k_i} m_a E_a}\big)$$

$$\mapsto \mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{Z_t}\otimes\mathcal{E}_{C-\sum_{a\in I_{k_i}} m_a E_a - \sum_{a<k_i} m_a E_a}\big)$$

and the vanishing of $\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{Z_t}\otimes\mathcal{E}_{C-\sum_{a\in I_{k_i}} m_a E_a - \sum_{a<k_i} m_a E_a}\big)$ by the fact that $Z_t \mapsto Y(\Gamma)$ is "at most relative dimension zero".

Once the surjectivity has been achieved, this surjection induces an isomorphism between the torsion free quotients of $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\Xi_{k_i}}\otimes\mathcal{E}_{C-\sum_{a\leq n} m_a E_a}\big)$, $\cong$ $\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{I}_{Z_t}\otimes\mathcal{E}_{C-\sum_{a\in I_{k_i}} m_a E_a - \sum_{a<k_i} m_a E_a}\big)$, with $\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}}\otimes\mathcal{E}_{C-\sum_{a\in I_{k_i}} m_a E_a - \sum_{a<k_i} m_a E_a}\big)$, by the argument in lemma 5. $\square$

The following lemma will be used frequently in the following discussion.

Lemma 10 *Let $\mathcal{G}_i, 0 \leq i \leq 4$ be five coherent sheaves over a smooth, connected and reduced scheme Y and let $\mathcal{G}_0 \mapsto \mathcal{G}_1 \mapsto \mathcal{G}_2 \mapsto \mathcal{G}_3 \mapsto \mathcal{G}_4$ be a sheaf exact sequence. Suppose that both of $\mathcal{G}_0$ and $\mathcal{G}_4$ are torsion sheaves and $(\mathcal{G}_2)_{torfree}$ is locally free. Suppose additionally that the induced morphism $(\mathcal{G}_1)_{torfree}\otimes k(y) \mapsto (\mathcal{G}_2)_{free}\otimes k(y)$ is injective for all the closed points $y \in Y$, then the torsion free quotients of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are all locally free and they form a short exact sequence of locally free sheaves,*

$$0 \mapsto (\mathcal{G}_1)_{free} \mapsto (\mathcal{G}_2)_{free} \mapsto (\mathcal{G}_3)_{free} \mapsto 0.$$

Proof: It is well known that any morphism from a torsion sheaf to a torsion free sheaf is trivial and any morphism from a torsion free sheaf to a torsion sheaf cannot be injective. By taking the double-duals of the original sequence, we get

$$\begin{array}{ccccc} \mathcal{G}_1 & \mapsto & \mathcal{G}_2 & \mapsto & \mathcal{G}_3 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G}_1^{**} & \mapsto & \mathcal{G}_2^{**} & \mapsto & \mathcal{G}_3^{**} \end{array}$$

The second row is acyclic and it induces an acyclic sequence on the torsion free parts $(\mathcal{G}_i)_{torfree}$, $1 \leq i \leq 3$,

$$(\mathcal{G}_1)_{torfree} \mapsto (\mathcal{G}_2)_{torfree} \mapsto (\mathcal{G}_3)_{torfree} \mapsto 0.$$

The above sequence is right-exact because by the commutative diagram $(\mathcal{G}_2)_{tor}$ is in the kernel of the composite surjection $\mathcal{G}_2 \mapsto \mathcal{G}_3 \mapsto (\mathcal{G}_3)_{torfree}$.

To show that it is also left exact and exact in the middle, notice that the acyclicity of the above sequence implies the surjection $(\mathcal{G}_2)_{torfree}/Im((\mathcal{G}_1)_{torfree}) \mapsto (\mathcal{G}_3)_{torfree} \mapsto 0$.

Consider the sequence

$$(\mathcal{G}_1)_{torfree} \mapsto (\mathcal{G}_2)_{torfree} \mapsto (\mathcal{G}_2)_{torfree}/Im((\mathcal{G}_1)_{torfree}) \mapsto 0.$$

By the assumption of our lemma, $(\mathcal{G}_1)_{torfree} \otimes k(y) \mapsto (\mathcal{G}_2)_{torfree} \otimes k(y)$ is injective for all closed points y. Because both $(\mathcal{G}_1)_{torfree}$ and $(\mathcal{G}_2)_{torfree}/Im((\mathcal{G}_1)_{torfree})$ are coherent, by exercise II.5.8(a) of [Ha], both $rank_{k(y)}(\mathcal{G}_1)_{torfree} \otimes k(y)$ and $rank_{k(y)}\big((\mathcal{G}_2)_{torfree}/Im((\mathcal{G}_1)_{torfree})\big) \otimes k(y) = rank_{k(y)}\big((\mathcal{G}_2)_{torfree} \otimes k(y)/Im((\mathcal{G}_1)_{torfree} \otimes k(y))\big)$ are upper semi-continuous. But by assumption $(\mathcal{G}_2)_{torfree}$ is known to be locally free, so by exercise II.5.8(b) of [Ha] and the connectivity of Y, $rank_{k(y)}(\mathcal{G}_2)_{torfree} \otimes k(y)$ is constant throughout the connected scheme Y. This forces $rank_{k(y)}(\mathcal{G}_1)_{torfree} \otimes k(y)$ and $rank_{k(y)}(\mathcal{G}_2)_{torfree}/Im((\mathcal{G}_1)_{torfree}) \otimes k(y) = rank_{k(y)}(\mathcal{G}_2)_{free} \otimes k(y) - rank_{k(y)}(\mathcal{G}_1)_{torfree} \otimes k(y)$ to be lower semi-continuous and to be constant. Therefore by exercises 3.17, 5.7-5.8 of chapter II of [Ha], $(\mathcal{G}_1)_{torfree}$ and the quotient $(\mathcal{G}_2)_{torfree}/Im((\mathcal{G}_1)_{torfree})$ are also locally free. In particular, $(\mathcal{G}_1)_{torfree} \mapsto (\mathcal{G}_2)_{torfree}$ induces a bundle injection [33] and it has to be a sheaf injection.

On the other hand, the surjection

$$(\mathcal{G}_2)_{torfree}/Im((\mathcal{G}_1)_{torfree}) \mapsto (\mathcal{G}_3)_{torfree} \mapsto 0$$

implies that $(\mathcal{G}_3)_{torfree}$ is the quotient of a locally free sheaf of the same generic rank. This implies that the kernel sheaf of this surjection must be a torsion sheaf. As there is no sheaf injection from a torsion sheaf into a locally free sheaf, the surjection is in fact a sheaf isomorphism and thus $(\mathcal{G}_3)_{torfree}$ is also locally free.

So we may replace the sheaves in the original short exact sequence

$$0 \mapsto (\mathcal{G}_1)_{torfree} \mapsto (\mathcal{G}_2)_{torfree} \mapsto (\mathcal{G}_2)_{torfree}/Im((\mathcal{G}_1)_{torfree}) \mapsto 0$$

by $(\mathcal{G}_1)_{free}$, $(\mathcal{G}_2)_{free}$, and $(\mathcal{G}_3)_{free}$, respectively, and the proof of this lemma is complete. □

In the following proposition we prove the locally freeness of the sheaf $\mathcal{V}_{quot} = (\mathcal{R}^1\pi_*(\mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}} \otimes \mathcal{E}_{C-\mathbf{M}(E)E}))_{torfree}$ and identify the equivalence class of $\mathcal{V}_{quot}$ in the K group explicitly.

[33] Because of the injectivity of the $\otimes k(y)$ version of morphisms.

Proposition 5 *The torsion free quotient of the coherent sheaf* $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}}\otimes \mathcal{E}_{C-\mathbf{M}(E)E}\big)$ *is locally free and it is equivalent to the direct sum of locally free sheaf* $\oplus_{1\leq l\leq p}\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_l}}\otimes\mathcal{E}_{C-\sum_{b\in I_{k_l}} m_bE_b-\sum_{p\geq d>l} e_{k_d}}\big)\otimes\tilde{\pi}^*\mathcal{O}(-\sum_{1\leq a<k_l} m_aE_{a;k_l})$ *in* $K_0(Y(\Gamma)\times T(M))$.

Proof of the proposition: For $p=1$, the sum of the divisors $\sum_{i\leq p}\Xi_{k_i}$ collapses to a single Ξ_{k_1}. The conclusion of locally freeness and the identity in $K_0(Y(\Gamma)\times T(M))$ are direct consequences of lemma 9. We prove the general case based on induction upon p.

By induction hypothesis, we know that the "locally free" quotient of $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{1\leq i\leq p-1}\Xi_{k_i}}\otimes \mathcal{E}_{C-\mathbf{M}(E)E-e_{k_p}}\big)$ is equivalent to $\oplus_{1\leq l\leq p-1}\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_l}}\otimes\mathcal{E}_{C-\sum_{b\in I_{k_l}} m_bE_b-\sum_{p\geq d>l} e_{k_d}}\big)\otimes \tilde{\pi}^*\mathcal{O}(-\sum_{1\leq a<k_l} m_aE_{a;k_l})$. To prove the $p-$th version of our proposition it suffices to prove the locally freeness of the torsion free sheaf and then prove the existence of a short exact sequence of locally free sheaves,

$$0\mapsto (\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{i\leq p-1}\Xi_{k_i}}(-\Xi_{k_p})\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\big))_{free}\mapsto (\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\big))_{free}$$
$$\mapsto (\mathcal{R}^1\pi_*\big(\mathcal{O}_{\Xi_{k_p}}\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\big))_{free}\mapsto 0.$$

By pushing forward the short exact sequence

$$0\mapsto \mathcal{O}_{\sum_{1\leq i\leq p-1}\Xi_{k_i}}(-\Xi_{k_p})\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\mapsto \mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}}\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\mapsto \mathcal{O}_{\Xi_{k_p}}\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\mapsto 0,$$

we get a long exact sequence

$$\mathcal{R}^0\pi_*\big(\mathcal{O}_{\Xi_{k_p}}\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\big)\mapsto \mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{1\leq i\leq p-1}\Xi_{k_i}}\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a-e_{k_p}}\big)$$
$$\mapsto \mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\big)\mapsto \mathcal{R}^1\pi_*\big(\mathcal{O}_{\Xi_{k_p}}\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\big)\mapsto 0.$$

The above sequence is right exact because $\sum_{1\leq i\leq p-1}\Xi_{k_i}\mapsto Y(\Gamma)$ is of relative dimension one over the base $Y(\Gamma)$ and so the corresponding second derived image sheaf along $\sum_{1\leq i\leq p-1}\Xi_{k_i}\mapsto Y(\Gamma)$ vanishes.

Because of the degree constraint on $\sum_{1\leq a\leq n} m_aE_a$, $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\Xi_{k_p}}\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\big)$ vanishes on the Zariski-open subset $Y_\Gamma\times T(M)\subset Y(\Gamma)\times T(M)$ and is a torsion sheaf over $Y(\Gamma)\times T(M)$. By lemma 9 and by the induction hypothesis the torsion free sheaves $(\mathcal{R}^1\pi_*\big(\mathcal{O}_{\Xi_{k_p}}\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\big))_{torfree}$ and $(\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{1\leq i\leq p-1}\Xi_{k_i}}\otimes \mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a-e_{k_p}}\big))_{torfree}$ are known to be locally free.

Then the desired short exact sequence of locally free sheaves is constructed from the acyclic sequence formed by the torsion free quotients of the above long

exact sequence, after we have shown the torsion free quotient of the middle factor $\mathcal{V}_{quot}$ is locally free.

The rest of the proposition is devoted to derive the locally freeness of $(\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}}\otimes \mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\big))_{torfree}$ and the exactness of the above acyclic sequence.

Step I: The locally freeness of the torsion free quotient (part).

The invertible sheaf $\mathcal{O}_{M_{n+1}}(-\sum_{1\leq a\leq n} m_aE_a)$ pulls back to an invertible sheaf on $\sum_{1\leq i\leq p}\Xi_{k_i}$, denoted by $\mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}}(-\sum_{1\leq a\leq n} m_aE_a)$. The invertible sheaf fails to be a sub-sheaf of $\mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}}$, therefore it is not an ideal sheaf on $\sum_{1\leq i\leq p}\Xi_{k_i}$. We point out the main cause. Let P be the union of $\{k_1, k_2, k_3, \cdots, k_p\}$ union with their direct and indirect ascendents in Γ. The canonically defined sheaf morphism $\mathcal{O}_{M_{n+1}}(-\sum_{a\in P} m_aE_a)\mapsto \mathcal{O}_{M_{n+1}}$ vanishes on the whole sub-scheme $\sum_{1\leq i\leq p}\Xi_{k_i}\subset M_{n+1}$ because the sections defining the divisors E_a, $a\in P$ vanish [34] on Ξ_{k_i} for all $1\leq i\leq p$. Thus, the defining section of $\mathcal{O}_{M_{n+1}}(-\sum_{1\leq a\leq n} m_aE_a)\mapsto \mathcal{O}_{M_{n+1}}$ vanishes on $\sum_{1\leq i\leq p}\Xi_{k_i}$ as well.

Consider the fiber product (intersection) of $\sum_{1\leq a\leq n} m_aE_a\subset M_{n+1}$ with $\sum_{1\leq i\leq p}\Xi_{k_i}\subset M_{n+1}$. Even though the fiber product is not a sub-scheme of $\sum_{1\leq i\leq p}\Xi_{k_i}$, it still contains a maximal sub-scheme Z as a divisor in $\sum_{1\leq i\leq p}\Xi_{k_i}$. Since Z is a divisor, the ideal sheaf defining Z, $\mathcal{I}_Z$ is locally free $\cong \mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}}(-Z)$.

Then we may write $\mathcal{O}_{\sum_{1\leq i\leq p}\Xi_{k_i}}(-\sum_{1\leq a\leq n} m_aE_a)$ as $\mathcal{I}_Z\otimes\mathcal{J}$, with $\mathcal{J}$ being locally free.

Tensoring the defining short exact sequence $0\mapsto\mathcal{I}_Z\mapsto\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\mapsto \mathcal{O}_Z\mapsto 0$ by $\mathcal{J}_C=\mathcal{J}\otimes\mathcal{E}_C$ and taking the derived long exact sequence along $\pi:\sum_{i\leq p}\Xi_{k_i}\mapsto Y(\Gamma)$, we find

$$\mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes\mathcal{J}_C\big)\mapsto\mathcal{R}^0\pi_*\big(\mathcal{O}_Z\otimes\mathcal{J}_C\big)$$

$$\overset{\delta}{\mapsto}\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\big)\mapsto\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes\mathcal{J}_C\big).$$

We know that $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes\mathcal{J}_C\big)$ is a torsion sheaf, since $deg_{\Xi_{k_i}/Y(\Gamma)}\mathcal{J}_C = m_i>0$ by a calculation shown on page 34.

Now we get the following short exact sequence on the torsion free quotients of the above sequence (based on lemma 10),

$$0\mapsto(\mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes\mathcal{J}_C\big))_{free}\mapsto(\mathcal{R}^0\pi_*\big(\mathcal{O}_Z\otimes\mathcal{J}_C\big))_{free}$$

$$\mapsto(\mathcal{R}^1\pi_*\big(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes\mathcal{E}_{C-\sum_{1\leq a\leq n} m_aE_a}\big))_{free}\mapsto 0,$$

once one shows (i). $(\mathcal{R}^0\pi_*\big(\mathcal{O}_Z\otimes\mathcal{J}_C\big))_{torfree}$ is locally free.

[34] This is why we want P to contain k_is or their ascendents. when i is a ascendent of j in Γ, the defining section of E_i is divisible by the defining section of E_j above $Y(\Gamma)$.

(ii). the injectivity $(\mathcal{R}^0\pi_*(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes\mathcal{J}_C))_{torfree}\otimes k(y)\mapsto(\mathcal{R}^0\pi_*(\mathcal{O}_Z\otimes\mathcal{J}_C))_{torfree}\otimes k(y)$ for all the closed points $y\in Y(\Gamma)\times T(M)$.

The proof of condition (i): Take [35] Z_f to be the union of the irreducible components of the divisor $Z\subset\sum_{1\leq i\leq p}\Xi_{k_i}$ which dominates $Y(\Gamma)$. Take Z_t be the union of the irreducible components of divisors in Z which do not dominate $Y(\Gamma)$. Because Z is the divisor induced by $\sum_{1\leq a\leq n}m_aE_a$, for a fixed i with $1\leq i\leq p$, the restriction of $\sum_{j_{k_i}}m_{j_{k_i}}E_{j_{k_i}}$ to Ξ_{k_i} defines a sub-scheme in Ξ_{k_i}, a union (with multiplicities) of cross-sections of $\Xi_{k_i}\mapsto Y(\Gamma)$. So the map $Z_f\mapsto Y(\Gamma)$ is a finite morphism. On the other hand, $Z_t\subset\sum_{1\leq i\leq p}\Xi_{k_i}$ maps onto a union of divisors in $Y(\Gamma)$. Because $Z=Z_f\cup Z_f$ is a union of divisors, we have the following short exact sequence of divisors in $\sum_{i\leq p}\Xi_{k_i}$,

$$0\mapsto\mathcal{O}_{Z_f}(-Z_t)\mapsto\mathcal{O}_Z\mapsto\mathcal{O}_{Z_t}\mapsto 0.$$

Since $Z_f\mapsto Y(\Gamma)$ is a finite morphism, $\mathcal{R}^1\pi_*(\mathcal{O}_{Z_f}(-Z_t)\otimes\mathcal{J}_C)$ vanishes and $\mathcal{R}^0\pi_*(\mathcal{O}_{Z_f}(-Z_t)\otimes\mathcal{J}_C)$ is locally free[36]. So the $\mathcal{J}_C$ twisted derived long exact sequence of the above short exact sequence truncates to a sheaf short exact sequence

$$0\mapsto\mathcal{R}^0\pi_*(\mathcal{O}_{Z_f}(-Z_t)\otimes\mathcal{J}_C)\mapsto\mathcal{R}^0\pi_*(\mathcal{O}_Z\otimes\mathcal{J}_C)\mapsto\mathcal{R}^0\pi_*(\mathcal{O}_{Z_t}\otimes\mathcal{J}_C)\mapsto 0.$$

Because Z_t is mapped into a proper sub-scheme of $Y(\Gamma)$ under π, its intersection with the generic fibers of $\pi:\sum_{i\leq p}\Xi_{k_p}\mapsto Y(\Gamma)$ must be empty. Thus $\mathcal{R}^0\pi_*(\mathcal{O}_{Z_t}\otimes\mathcal{J}_C)$ is a torsion sheaf.

One the other hand, it is easy to check [37] that $\mathcal{R}^0\pi_*(\mathcal{O}_{Z_f}(-Z_t)\otimes\mathcal{J}_C)\otimes k(y)\mapsto\mathcal{R}^0\pi_*(\mathcal{O}_Z\otimes\mathcal{J}_C)\otimes k(y)$ is injective for all the closed points y on $Y(\Gamma)\times T(M)$. lemma 6 the torsion free quotient of $\mathcal{R}^0\pi_*(\mathcal{O}_Z\otimes\mathcal{J}_C)$ is isomorphic to $\mathcal{R}^0\pi_*(\mathcal{O}_{Z_f}(-Z_t)\otimes\mathcal{J}_C)$ and is known to be locally free.

So we know that $(\mathcal{R}^0\pi_*(\mathcal{O}_Z\otimes\mathcal{J}_C))_{torfree}$ is locally free and the condition (i). has been proved.

The proof of condition (ii): By the derivation of condition (i), the original condition (ii) is equivalent to the injectivity $(\mathcal{R}^0\pi_*(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes\mathcal{J}_C))_{torfree}\otimes k(y)\mapsto\mathcal{R}^0\pi_*(\mathcal{O}_{Z_f}(-Z_t)\otimes\mathcal{J}_C)\otimes k(y)$ for all the closed points $y\in Y(\Gamma)\times T(M)$. On the other hand, the $k(y)$-twisted zero-th derived image sheaves map into the zero-th fiberwise sheaf cohomologies injectively[38], so it suffices to check the

[35]The subscripts f and t of Z_f and Z_t correspond to the keywords "free" and "torsion" as they are closely related to the locally free part and the torsion part of $\mathcal{R}^0\pi_*(\mathcal{O}_Z\otimes\mathcal{J}_C)$.

[36]Its rank is nothing but the relative length of $Z_f\mapsto Y(\Gamma)$.

[37]By comparing with the corresponding short exact sequence of the fiber above y. Consult the next argument below and the next footnote for a similar argument.

[38]This follows from taking the global sections of a twisted version of the exact sequence $0\mapsto\mathcal{I}_{\sum_{i\leq p}\Xi_{k_i}|_y}\mapsto\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\mapsto\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}|_y}\mapsto 0$ for all $y\in Y(\Gamma)$.

following injection $H^0(\sum_{i\le p}\Xi_{k_i}\times_{Y(\Gamma)\times T(M)}\{y\},\mathcal{J}_C|_y)\mapsto H^0(Z_f\times_{Y(\Gamma)\times T(M)}\{y\},\mathcal{O}_{Z_f}(-Z_t)\otimes\mathcal{J}_C|_y)$ for all the closed points y in $Y(\Gamma)\times T(M)$. By composing with the natural morphism induced by $\mathcal{O}_{Z_f}(-Z_t)\mapsto\mathcal{O}_{Z_f}$, it suffices to check the injectivity of $H^0(\sum_{i\le p}\Xi_{k_i}\times_{Y(\Gamma)\times T(M)}\{y\},\mathcal{J}_C|_y)\mapsto H^0(Z_f\times_{Y(\Gamma)\times T(M)}\{y\},\mathcal{O}_{Z_f}\otimes\mathcal{J}_C|_y)$. But this map is nothing but the restriction map of fiberwise global sections above y to the finite sub-scheme $Z_f\times_{Y(\Gamma)\times T(M)}\{y\}$. If this map is not injective, there must be a non-zero global section of $\mathcal{J}_C|_y$ on the fiber $\sum_{i\le p}\Xi_{k_i}\times_{Y(\Gamma)\times T(M)}\{y\}$ which vanishes along the finite sub-scheme $Z_f\times_{Y(\Gamma)\times T(M)}\{y\}$. In particular, we may restrict this fiberwise global section to one particular $\Xi_{k_l}\times_{Y(\Gamma)\times T(M)}\{y\}$ (for some $1\le l\le p$) over which the section does not vanish identically. We derive the contradiction by computing the degree of the invertible sheaf in two different ways.

Consider the index subset $P\subset\{1,2,\cdots,n\}$ collecting k_l and all its direct or indirect ascendents in Γ.

On the one hand, the degree of the invertible sheaf [39], $deg_{\Xi_{k_l}\times_{Y(\Gamma)\times T(M)}\{y\}}\mathcal{J}_C|_y$, is $deg_{\Xi_{k_l}\times_{Y(\Gamma)\times T(M)}\{y\}}\mathcal{O}_{\sum_{1\le i\le p}\Xi_{k_i}}(-\sum_{a\in P}m_aE_a)=e_{k_l}\cdot(-\sum_{a\in P}m_aE_a)=(E_{k_l}-\sum_{j_{k_l}}E_{j_{k_l}})\cdot(-\sum_{a\in P}m_aE_a)=-m_{k_l}E_{k_l}^2=m_{k_l}$ (since $k_l\in P$ and the descendent j_{k_l} of k_l can never be an ascendent of k_l, any term of the form $(-E_{j_{k_l}})\cdot(-m_aE_a)$ contributes trivially to the sum). On the other hand, the same degree must be no less than the relative length, $length((Z_f\cap\Xi_{k_l})\times_{Y(\Gamma)\times T(M)}\{y\})$, of the finite scheme $(Z_f\cap\Xi_{k_l})\times_{Y(\Gamma)\times T(M)}\{y\}$ along which the section vanishes. However the length of this finite scheme is nothing but the sum of multiplicities for all $m_{j_{k_l}}E_{j_{k_l}}$ along[40] Ξ_{k_l} and is equal to $\sum_{j_{k_l}}m_{j_{k_l}}$.

Combining these observations together we get an inequality $m_{k_l}\ge\sum_{j_{k_l}}m_{j_{k_l}}$, which implies $e_{k_l}\cdot(C-\mathbf{M}(E)E)\ge 0$. This violates our choices of e_{k_l} of making $e_{k_l}\cdot(C-\mathbf{M}(E)E)<0$! So the original $k(y)$-vector space morphisms must be injective for all y. The proof of condition (ii). is finished.

Step II: The proof of exactness of the acyclic sequence.

We plan to derive it by adopting the commutative diagram chasing argument.

Recall that the inclusion $\sum_{i\le p-1}\Xi_{k_i}\subset\sum_{i\le p}\Xi_{k_i}$ of $\mathbf{P}^1$ fibrations (removing the last Ξ_{k_p}) induces an inverse image of the ideal sheaf $\mathcal{I}_Z$ in $\sum_{i\le p-1}\Xi_{k_i}$, denoted by $\mathcal{I}_{Z'}$. Then the restriction of Z, Z', can be viewed as a sub-scheme of Z by removing the intersection $Z''=\Xi_{k_p}\cap Z$. Similarly, both Z_f and Z_t are restricted to Z'_f and Z'_t, respectively and likewise we also have $Z''=Z''_f+Z''_t=\Xi_{k_p}\cap Z_f+\Xi_{k_p}\cap Z_t$. To summarize, we have the following commutative diagram of three rows of short exact sequences,

[39] The degree can be calculated along any smooth fiber of Ξ_{k_l} over Y_Γ. moreover it is easy to see that the inverse image of the invertible sheaf $\mathcal{J}$ to the sub-locus $\Xi_{k_l}\times_{M_n}Y_\Gamma$ is isomorphic to $\mathcal{O}_{\Xi_{k_l}\times_{M_n}Y_\Gamma}(-\sum_{a\in P}m_aE_a)$ over Y_Γ. it is because the defining section of E_a, $a\in P$ vanishes on $\Xi_{k_i}|_y$ over y.

[40] The index j_{k_l} is a typical direct descendent index of k_l in the admissible graph Γ.

$$\begin{array}{ccccc}
\mathcal{O}_{Z'_f}(-Z'_t - Z'') & \mapsto & \mathcal{O}_{Z'}(-Z'') & \mapsto & \mathcal{O}_{Z'_t}(-Z'') \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{Z_f}(-Z_t) & \mapsto & \mathcal{O}_{Z} & \mapsto & \mathcal{O}_{Z_t} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{Z''_f}(-Z''_t) & \mapsto & \mathcal{O}_{Z''} & \mapsto & \mathcal{O}_{Z''_t}
\end{array}$$

All of the three rows and the first two columns in this above diagram are short exact sequences constructed from tensoring invertible sheaves with a divisorial exact sequence of the following form, $0 \mapsto \mathcal{O}_B(-A) \mapsto \mathcal{O}_{A+B} \mapsto \mathcal{O}_B \mapsto 0$.

Moreover we have the following commutative diagram [41] (after substituting by the short hand notations $\underline{C} = C - \sum_{1\leq a\leq n} m_a E_a = C - \mathbf{M}(E)E$).

$$\begin{array}{ccccc}
(\mathcal{R}^0\pi_*(\mathcal{O}_{\sum_{i\leq p-1}\Xi_{k_i}}(-\Xi_{k_p})\otimes \mathcal{J}_C))_{free} & \mapsto & \mathcal{R}^0\pi_*(\mathcal{O}_{Z'_f}(-Z'_t - Z'')\otimes \mathcal{J}_C) & \overset{\delta}{\mapsto} & (\mathcal{R}^1\pi_*(\mathcal{O}_{\sum_{i\leq p-1}\Xi_{k_i}}\otimes \mathcal{E}_{\underline{C}-e_{k_p}}))_{free} \\
\downarrow & & \downarrow & & \downarrow \\
(\mathcal{R}^0\pi_*(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes \mathcal{J}_C))_{free} & \mapsto & \mathcal{R}^0\pi_*(\mathcal{O}_{Z_f}(-Z_t)\otimes \mathcal{J}_C) & \overset{\delta}{\mapsto} & (\mathcal{R}^1\pi_*(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes \mathcal{E}_{\underline{C}}))_{free} \\
\downarrow & & \downarrow & & \downarrow \\
(\mathcal{R}^0\pi_*(\mathcal{O}_{\Xi_{k_p}}\otimes \mathcal{J}_C))_{free} & \mapsto & \mathcal{R}^0\pi_*(\mathcal{O}_{Z''_f}(-Z''_t)\otimes \mathcal{J}_C) & \overset{\delta}{\mapsto} & (\mathcal{R}^1\pi_*(\mathcal{O}_{\Xi_{k_p}}\otimes \mathcal{E}_{\underline{C}}))_{free}
\end{array}$$

$\diamondsuit$ We claim that all three rows and the first two columns in this commutative diagram are short exact sequences: By the earlier discussion based on lemma 10, the sheaves in the second rows are all locally free (this justifies the usage of the subscript $(\cdot)_{free}$ above). In the same argument the second row has been shown to be short exact. We may adopt a parallel argument on $\sum_{i\leq p-1}\Xi_{k_i}$ applied to $Z' = Z'_f \cup Z'_t$ or on Ξ_{k_p} applied to $Z'' = Z''_f \cup Z''_t$, thus the sheaves in the first and the third rows are all locally free and both of the first and the third rows are short exact as well. The first column is the locally (torsion) free summand of a derived long exact sequence, it is exact based on lemma 10 and we argue as the following: The locally freeness of the factor in the middle is already known. The injectivity of the $\bullet \otimes k(y)$ version of the first column in the above diagram

$$(\mathcal{R}^0\pi_*(\mathcal{O}_{\sum_{i\leq p-1}\Xi_{k_i}}(-\Xi_{k_p})\otimes \mathcal{J}_C))_{free}\otimes k(y) \mapsto (\mathcal{R}^0\pi_*(\mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}}\otimes \mathcal{J}_C))_{free}\otimes k(y)$$

is a direct consequence of the injectivity of the fiberwise zero-th cohomologies,

$$H^0(\sum_{i\leq p-1}\Xi_{k_i}\times_{Y(\Gamma)\times T(M)}\{y\}, \mathcal{O}(-\Xi_{k_p}\times_{Y(\Gamma)\times T(M)}\{y\})\otimes \mathcal{J}_C|_y) \mapsto H^0(\sum_{i\leq p}\Xi_{k_i}\times_{Y(\Gamma)\times T(M)}\{y\}, \mathcal{J}_C|_y).$$

[41] We have skipped the inverse image notation for $\mathcal{J}_C$ to the various sub-schemes, in order to make the formula less complicated.

The desired injectivity of the H^0 morphism has been the direct consequence of the following short exact sequence on the fiber above y,

$$0 \mapsto \mathcal{O}_{\sum_{i\leq p-1}\Xi_{k_i}\times_{Y(\Gamma)\times T(M)}\{y\}}(-\Xi_{k_p}\times_{Y(\Gamma)\times T(M)}\{y\}) \mapsto \mathcal{O}_{\sum_{i\leq p}\Xi_{k_i}\times_{Y(\Gamma)\times T(M)}\{y\}} \mapsto \mathcal{O}_{\Xi_{k_p}\times_{Y(\Gamma)\times T(M)}\{y\}} \mapsto 0.$$

So the exactness of the first column is ensured.

Finally the second column is short exact as it is the derived short exact sequence (remembering that $Z'_f \mapsto Y(\Gamma)$ is a finite morphism) of a twisted short exact sequence in the the first column of the previous commutative diagram on Z, Z_f, Z_t, Z'', Z''_f, Z''_t, Z', Z'_f, Z'_t.

The third column has been known to be acyclic. Then its exactness follows from the standard diagram-chasing technique. □

At the end of the subsection, we define a short-hand notation,

Definition 4 *Define the locally free sheaf $\tilde{\mathcal{V}}_{quot}$ to be*

$$\tilde{\mathcal{V}}_{quot} = \oplus_{1\leq l\leq p}\mathcal{R}^1\tilde{\pi}_*(\mathcal{O}_{\tilde{\Xi}_{k_l}}\otimes\mathcal{E}_{C-\sum_{b\in I_{k_l}} m_bE_b-\sum_{p\geq d>l} e_{k_d}})\otimes\tilde{\pi}^*\mathcal{O}(-\sum_{1\leq a<k_l} m_aE_{a;k_l}).$$

Following our convention, the corresponding vector bundle will be denoted by $\tilde{\mathbf{V}}_{quot}$.

4 The Localized Chern Classes and the Discrepancy of the Top Chern Classes

Let us consider the following general set up. Let X be [42] a purely m dimensional scheme and let $\mathbf{E} \mapsto X$, $F \mapsto X$ be vector bundles over X of the same rank, say e, and let $\sigma : \mathbf{E} \mapsto \mathbf{F}$ be a bundle homomorphism on X exact off a closed subset Z. Then in principle the difference of Chern classes of $\mathbf{F}$ and $\mathbf{E}$ should be expressible as cycle classes 'localized' in Z. In particular, when $Z = \emptyset$, the map σ induces an isomorphism between $\mathbf{F}$ and $\mathbf{E}$ and their Chern classes coincide.

For the convenience of the reader, we review the construction of [F], page 348 (c). as a special case of the graph construction of MacPherson. We have changed a few notations from the original notations of [F].

Proposition 6 *Let $cl(\mathbf{E}), cl(\mathbf{F})$ denote a polynomial of Chern classes of $\mathbf{E}$ and $\mathbf{F}$, respectively. Let $G = Grass_e(\mathbf{E}\oplus\mathbf{F})$ be the e-plane Grassmanian bundle over X and let $\zeta \mapsto G$ be the universal rank e bundle over G. Then there exists a cycle $\sum_{i\geq 1} n_iV_i \subset G$ supporting over Z, $\eta_i : V_i \mapsto Z$ the projection map, such that*

[42] The notation X used in this section is not the same X used in the the proof of our main theorem.

$$cl(\mathbf{E}) \cap [X] - cl(\mathbf{F}) \cap [X] = \sum_{V_i \mapsto Z} n_i \eta_{i*}\{cl(\zeta) \cap [V_i]\}.$$

Define $\phi : X \times \mathbf{A} \mapsto G \times \mathbf{P}^1$ by sending (x,t) to the graph of $t\sigma(x) \times (1,t)$. Then define W to be the closure of the image in $G \times \mathbf{P}^1$. Set $W_\infty = i_\infty^*[W] = \sum_{i \geq 0} n_i V_i$ to be the fiber of $W \mapsto \mathbf{P}^1$ above $\{\infty\} \subset \mathbf{P}^1$.

Because σ is bijective on $X - Z$, there exists a special component, say V_0, with $n_0 = 1$, and is birational to X through the projection $G \mapsto X$. In fact it is isomorphic to X. Excluding this component from W_∞, it turns out that the remaining $\sum_{i>0} n_i V_i$, $\eta_i : V_i \mapsto Z$ has the desired property. For the full details, consult page 340-348, section 18.1 the graph construction, of [F].

The graph construction allows us to express the difference of characteristic classes of $\mathbf{E}$ and $\mathbf{F}$ by cycle classes in Z, constructed from σ through the limiting process. In particular, it implies the following identity

$$\{c_e(\mathbf{E}) - c_e(\mathbf{F})\} \cap [X] = \sum_{V_i \mapsto Z} n_i \eta_{i*}\{c_e(\eta) \cap [V_i]\}.$$

But it might be technical to identify these $[V_i]$ explicitly.

Suppose that one is given additionally a global section $s_0 \in \Gamma(X, \mathbf{E})$, then $s = \sigma(s_0) \in \Gamma(X, \mathbf{F})$ is a global section of $\mathbf{F}$.

The localized top Chern class construction on page 244, section 14.1 of [F] defines localized classes $\mathbf{Z}(s_0) \in \mathcal{A}_{m-e}(Z(s_0))$, $\mathbf{Z}(s) \in \mathcal{A}_{m-e}(Z(s))$ and their push-forward into X are equal to $c_e(\mathbf{E}), c_e(\mathbf{F})$, respectively.

Thus the datum of the sections $s_0, s = \sigma(s_0)$ may be used to express the difference $c_e(\mathbf{E}) - c_e(\mathbf{F})$ as geometric cycles relating to s_0 and σ.

The first hint to such a possibility is the following proposition,

Proposition 7 *Let $\sigma : \mathbf{E} \mapsto \mathbf{F}$, X, $s_0 : X \mapsto \mathbf{E}$ be as above and let $s_{\mathbf{E}} : X \mapsto \mathbf{E}$ denote the zero section of $\mathbf{E}$ and let $\pi_{\mathbf{E}} : \mathbf{E} \mapsto X$ be the bundle projection, then the kernel $Ker(\sigma)$ determines an algebraic sub-cone $\mathbf{C}_\rho$ of the total space of $\mathbf{E}$ and there exists an scheme theoretical equality $Z(s) = Z(s_0) \cup \pi_{\mathbf{E}}((\mathbf{C}_\rho - s_{\mathbf{E}}(X)) \cap s_0(X))$ between the zero loci.*

In general we may write $\mathbf{C}_\rho = \cup_{i \geq 0} \mathbf{C}_{\rho_i}$, where $\mathbf{C}_{\rho_0}$ is the zero section cone $s_{\mathbf{E}}(X)$ and $\cup_{i>0} \mathbf{C}_{\rho_i}$ is the union of the remaining irreducible components supporting inside Z. Because the proposition is parallel to the discussion in proposition 12 in [Liu5], we only give a sketch of the proof:

Sketch of the proof: Let $\mathcal{E}$, $\mathcal{F}$ be the locally free sheaves over X associated to $\mathbf{E}$, $\mathbf{F}$, respectively.

The sheaf morphism $\mathcal{E} \mapsto \mathcal{F}$ induces a dual morphism $\mathcal{F}^* \mapsto \mathcal{E}^*$ with cokernel sheaf $\mathcal{R}$.

Consider the $\mathcal{O}_X$ algebra $\mathbf{S}^\cdot$ generated by $\mathbf{S}^1 = \mathcal{R}$, then $\mathbf{C}_\rho = Spec(\mathbf{S}^\cdot)$ defines a sub-cone in the vector bundle cone of $\mathbf{E}$. By tensoring with $k(x)$

(which is right exact) for all $x \in X$ and taking the left exact contravariant functor $HOM_{k(x)}(\cdot, k(x))$, one may see easily that this cone is the kernel subcone of $\sigma : \mathbf{E} \mapsto \mathbf{F}$.

One may observe $Z(s) = Z(s_0) \cup \pi_{\mathbf{E}}((\mathbf{C}_\rho - s_{\mathbf{E}}(X)) \cap s_0(X))$ on the set theoretical level rather easily. The equality as schemes follows from a parallel discussion as in proposition 12/corollary 3 of [Liu5]. We omit the details here. □

One notices that besides the unique component $\mathbf{C}_{\rho_0}$ which is equal to the zero section $s_{\mathbf{E}}(X)$, the union of the remaining cones, $\mathbf{C}_\rho - s_{\mathbf{E}}(X) = \cup_{i>0}\mathbf{C}_{\rho_i}$, supports exactly on Z.

On the one hand, we have the following residual intersection theory formula on the (localized) top Chern class (see page 245, example 14.1.4. of [F])

Proposition 8 *Let $\mathbf{F} \mapsto X$ be a rank e vector bundle over a purely m-dimensional scheme X and let $s : X \mapsto \mathbf{F}$ be a global section. Let D be an effective Cartier divisor contained in $Z(s)$, then there exists a section s' of $\mathbf{F} \otimes \mathcal{O}(-D)$ such that*
(i). $\mathbf{F} \otimes \mathcal{O}(-D) \mapsto \mathbf{F}$ maps s' to s.
(ii). $\mathbf{Z}(s) = \mathbf{Z}(s') + \sum_{1 \leq i \leq e} (-1)^{i-1} c_{e-i}(\mathbf{F}) \cap D^{i-1} \cap [D]$.

It makes sense to combine proposition 6, 7 and 8 and unify these observations together.

Firstly, let $\sigma : \mathbf{E} \mapsto \mathbf{F}$ be isomorphic off $Z \subset X$ as before. Consider the subscheme $Z(s) \cap Z \subset X$. One may blow it up into an exceptional Cartier divisor, denoted as D in the blown up scheme $\tilde{X}$. From the general construction of blowing up coherent sheaves of ideals, (see page 163-169 of [Ha] and B.6 page 435-437 of [F]), D is isomorphic to the projectified normal cone $\mathbf{P}(\mathbf{C}_{Z(s)\cap Z} X)$. Then one may apply proposition 8 to $\tilde{X}$, $g^*\mathbf{F} \mapsto \tilde{X}$, where $g : \tilde{X} \mapsto X$ denotes the blowing down map with the exceptional divisor D.

The following simple lemma identifies the term $\sum_{1 \leq i \leq e} (-1)^{i-1} c_{e-i}(\mathbf{F}) \cap D^{i-1} \cap [D]$ for us,

Lemma 11 *The cycle class $g_*\{\sum_{1 \leq j \leq e} (-1)^{j-1} c_{e-j}(\mathbf{F}) \cap D^{j-1} \cap [D]\} \in \mathcal{A}_{m-e}(Z(s) \cap Z)$ is equal to the localized contribution of top Chern class of $Z(s) \cap Z$, $\mathbf{Z}_{Z(s)\cap Z}(s) = \{c(\mathbf{F}|_{Z(s)\cap Z}) \cap s(Z(s) \cap Z, X) \cap [X]\}_{m-e}$. (see definition 1 of in section 5 of [Liu5])*

Proof of the lemma: Recall from page 71 of [F] that the Segre class of a cone[43] $C \mapsto Y$ over Y is defined to be

$$s(C) = q_*(\sum_{j \geq 0} c_1(\mathcal{O}(1))^j \cap [\mathbf{P}(C \oplus 1)]) \in \mathcal{A}_*(Y),$$

where $q : \mathbf{P}(C \oplus 1) \mapsto Y$ is the projection map.

[43] Do not confuse this cone C with the cohomology class $C \in H^{1,1}(M, \mathbf{Z})$.

In our context we take C to be the normal cone $\mathbf{C}_{Z(s)\cap Z}X$ and $Y = Z(s)\cap Z$, and so $D = \mathbf{P}(C_{Z(s)\cap Z}X)$. On the one hand, $\mathbf{P}(\mathbf{C}_{Z(s)\cap Z}X)$ is a divisor in $\mathbf{P}(\mathbf{C}_{Z(s)\cap Z}X \oplus 1)$ defined by $c_1(\mathcal{O}(1))$. On the other hand, $D = \mathbf{P}(\mathbf{C}_{Z(s)\cap Z}X)$ is an exceptional divisor in $\tilde{X}$, thus $D = -c_1(\mathcal{O}(1))$ and we have

$$g_*\{\sum_{1\leq j\leq e}(-1)^{j-1}c_{e-j}(g^*\mathbf{F})\cap D^{j-1}\cap[D]\} = g_*\{\sum_{1\leq j\leq e}c_{e-j}(g^*\mathbf{F})\cap c_1(\mathcal{O}(1))^{j-1}\cap[\mathbf{P}(\mathbf{C}_{Z(s)\cap Z}X)]\}$$

$$= \{c(\mathbf{F}|_{Z(s)\cap Z}) \cap g_*(\sum_{1\leq j\leq e} c_1(\mathcal{O}(1))^j \cap [\mathbf{P}(\mathbf{C}_{Z(s)\cap Z}\oplus 1)])\}_{m-e}$$

$$= \{c(\mathbf{F}|_{Z(s)\cap Z})g_*(\sum_{1\leq j} c_1(\mathcal{O}(1))^j \cap [\mathbf{P}(\mathbf{C}_{Z(s)\cap Z}\oplus 1)])\}_{m-e}$$

$$= \{c(\mathbf{F}|_{Z(s)\cap Z}) \cap s(Z(s)\cap Z, X)\}_{m-e}.$$

□

The expression $\{c(\mathbf{F}|_{Z(s)\cap Z})\cap s(Z(s)\cap Z, X)\}_{m-e}$ is nothing but the localized contribution of the top Chern class $\mathbf{Z}_Z(s)$ discussed in section 6 of [Liu5].

Thus, the identity in proposition 8 (ii). can be re-written as $\mathbf{Z}(s) = \mathbf{Z}(s') + \mathbf{Z}_{Z(s)\cap Z}(s)$. When $\sigma : \mathbf{E} \mapsto \mathbf{F}$ is exact off Z, we compare it to the top Chern class identity

$$c_e(\mathbf{F}) \cap [X] = c_e(\mathbf{E}) \cap [X] - \sum_{V_i \mapsto Z} n_i \eta_{i*}\{cl(\zeta) \cap [V_i]\}.$$

Introducing the following equivalent relationship which will be essential to the invariant enumeration in section 6.

Definition 5 *Let η_1, η_2 with $\eta_1\cap, \eta_2\cap : \mathcal{A}_k(X) \mapsto \mathcal{A}_0(X)$ be two grade-k characteristic classes on an m dimensional complete scheme X. The classes η_1 and η_2 are said to be numerically equivalent, denoted as $\eta_1 \overset{n}{=} \eta_2$ if for all $\alpha \in \mathcal{A}_{m-k}(X)$, $\int_X \eta_1 \cap \alpha = \int_X \eta_2 \cap \alpha$. In other words, $\int_X \eta_1 \cap \cdot$ and $\int_X \eta_2 \cap \cdot$ define identical integral operations from $\mathcal{A}_{m-k}(X)$ to $\mathcal{A}_0(pt)$.*

It makes sense to ask the following question,

Question: Are $i_*\mathbf{Z}(s') \overset{n}{=} c_e(g^*\mathbf{E})\cap[\tilde{X}]$ and $\mathbf{Z}_{Z(s)\cap Z}(s) \overset{n}{=} -\sum_{V_i\mapsto Z} n_i\eta_{i*}\{c_e(\zeta)\cap [V_i]\}$?

The following proposition answers the question affirmatively. In order to apply the current discussion to the explicit enumeration problem in section 6, we generalize the setting slightly.

Let X be a purely m dimensional reduced complete scheme as above. $Z \subset X$ is a closed sub-scheme of X.

Proposition 9 *Let $\mathbf{E}, \mathbf{F}$ be two rank e vector bundles on X. Suppose that $\sigma : \mathbf{E} \mapsto \mathbf{F}$ is a bundle morphism on X isomorphic off Z, and that $s_0 : X \mapsto \mathbf{E}$ is a global section of $\mathbf{E}$ inducing the global section of $\mathbf{F}$, $s = \sigma(s_0) : X \mapsto \mathbf{F}$. According to proposition 7, there exists a union of irreducible cones $\cup_{i \in I = \{1,2\cdots,n\}} \mathbf{C}_{\rho_i}$ supported over Z such that $Z(s) = Z(s_0) \cup \cup_{i>0} \pi_{\mathbf{E}}(\mathbf{C}_{\rho_i} \cap s_0(X))$.*

Let $I = \coprod_{1 \leq p \leq r} I_p$ be a partition of the index set I into disjoint subsets $I_p \subset I$. Consider the r-consecutive scheme theoretical blowing ups of X along the strict transformations of $\cup_{i \in I_p} \pi_{\mathbf{E}}(\mathbf{C}_{\rho_i} \cap s_0(X))$, and denote the resulting blown up scheme by $\tilde{X}$. Let $f : \tilde{X} \mapsto X$ to be the r-compositions of blowing down projection maps.

Let $D = f^{-1}(Z(s) \cap Z) \subset \tilde{X}$ be the exceptional Cartier divisor in $\tilde{X}$. Let $s' : \tilde{X} : f^*\mathbf{F} \otimes \mathcal{O}(-D)$ be the residual section which maps to s through $f^*\mathbf{F} \otimes \mathcal{O}(-D) \mapsto f^*\mathbf{F}$. Let $i : Z(s') \mapsto \tilde{X}$ be the inclusion map, then

$$c_e(f^*\mathbf{E}) \cap [\tilde{X}] \overset{n}{=} i_* \mathbf{Z}(s') \in \mathcal{A}_{m-e}(\tilde{X}),$$

i.e. they define the same cap product operation from $\mathcal{A}_e(\tilde{X})$ to $\mathcal{A}_0(pt) \cong \mathbf{Z}$.

Even though s' is not directly related to s_0 and $\mathbf{E}$, the cycle $\mathbf{Z}(s')$ still defines a version of localized top Chern class "localized" in $Z(f^*s)$ away from D. The proposition implies that its image under the push-forward morphism i_* is numerically equivalent to $c_e(f^*\mathbf{E})$.

Proof of proposition 9: The main idea of the proof is to construct an ambient space containing $\tilde{X}$, some auxiliary vector bundles and sections which are used to relate both sides of the equality. Define $Y = \mathbf{P}(\mathbf{E} \oplus \mathbf{C})$ and let $\pi_Y : Y \mapsto X$ denote the projection map. Through the map $\mathbf{v} \mapsto (\mathbf{v}, 1)$ the total space of the vector bundle $\mathbf{E}$ can be viewed as an open subspace of Y, which is the complement of the closed hypersurface $\mathbf{P}(\mathbf{E}) \subset Y$ at infinity. Thus, $\mathbf{P}(\mathbf{E})$ can be viewed as the compactification at infinity of $\mathbf{E} \subset \mathbf{P}(\mathbf{E} \oplus \mathbf{C})$.

Consider the hyperplane line bundle on Y, denoted as $\mathbf{H}$. Then the projection map $\mathbf{E} \oplus \mathbf{C} \mapsto \mathbf{C}$ over X induces a section of $\mathbf{H}$ vanishing exactly at $\mathbf{P}(\mathbf{E})$. Then $[\mathbf{P}(\mathbf{E})] \in \mathcal{A}_\cdot(Y)$ is equal to $c_1(\mathbf{H}) \cap [Y]$. On the other hand, the zero section $s_{\mathbf{E}}(X)$ embedded in $\mathbf{E} \subset Y$ can be viewed as the zero locus of a canonical section of $\pi_Y^*\mathbf{E} \otimes \mathbf{H}$ determined by the bundle map $\mathbf{E} \oplus \mathbf{C} \mapsto \mathbf{E}$. Thus $[s_{\mathbf{E}}(X)] = c_e(\pi_Y^*\mathbf{E} \otimes \mathbf{H}) \cap [Y]$.

The composition of $\mathbf{E} \oplus \mathbf{C} \mapsto \mathbf{E}$ and $\sigma : \mathbf{E} \mapsto \mathbf{F}$ induces a tautological section $\underline{s}$ of $\pi_Y^*\mathbf{F} \otimes \mathbf{H}$ on Y. The following lemma characterizes its zero locus $Z(\underline{s})$.

Lemma 12 *Let $\mathbf{C}_\rho$ denote the algebraic sub-cone of $\mathbf{E}$ corresponding to $Ker(\sigma)$. Then $\mathbf{C}_\rho$ can be identified canonically with a locally closed sub-scheme of Y and $Z(\underline{s}) \subset Y$ is the closure of $\mathbf{C}_\rho$, $\mathbf{P}(\mathbf{C}_\rho \oplus 1)$, in Y.*

Proof of the lemma: A point in $Y = \mathbf{P}(\mathbf{E} \oplus \mathbf{C})$ is inside $Z(\underline{s})$ if and only if the corresponding ray in $\mathbf{E} \oplus \mathbf{C}$ maps to zero under $\mathbf{E} \oplus \mathbf{C} \mapsto \mathbf{E} \mapsto \mathbf{F}$. In other

words, when the ray is in the direction inside the cone $\mathbf{C}_\rho \oplus 1$ corresponding to the kernel of $\mathbf{E} \oplus \mathbf{C} \mapsto \mathbf{F}$. So $Z(\underline{s}) = \mathbf{P}(\mathbf{C}_\rho \oplus 1)$. $\square$

On the other hand, the space Y can be viewed as 1-plane Grassmanian bundle of $\mathbf{E} \oplus \mathbf{C}$ over X. Viewed as a universal object, one may use it as our playground to prove proposition 9.

Firstly write $\mathbf{C}_\rho$ as $\cup_{0 \leq i \leq n} \mathbf{C}_{\rho_i}$ with $\mathbf{C}_{\rho_0} = s_{\mathbf{E}}(X)$. Then $\cup_{i \in I} \mathbf{C}_{\rho_i} = \mathbf{C}_\rho - s_{\mathbf{E}}(X)$ is a union of irreducible sub-cones and $G = \mathbf{P}((\cup_{i \in I} \mathbf{C}_{\rho_i}) \oplus 1)$ defines a closed sub-scheme of Y.

Notice that $G = \cup_{i \in I} \mathbf{P}(\mathbf{C}_{\rho_i} \oplus 1)$ and we may set $G_l = \cup_{i \in I_l} \mathbf{P}(\mathbf{C}_{\rho_i} \oplus 1)$. Then we may write $G = \cup_{1 \leq l \leq r} G_l$. It is obvious that $Z(\underline{s}) = s_{\mathbf{E}}(X) \cup G$.

Secondly one blows up Y consecutively along (the strict transformations under previous blowing ups of) $G_p, 1 \leq p \leq r$, following the exactly the same blowing up orders to construct $\tilde{X}$ from X. Denote the resulting scheme [44] $\tilde{Y}$ and denote the union of the resulting exceptional Cartier divisors D_Y. Denote the composite blowing down map $\tilde{Y} \mapsto Y$ by f_Y. Because $G \subset Z(\underline{s})$ and the sub-scheme G has been blown up consecutively to get $\tilde{Y}$, the pull-back section $(f_Y)^* \underline{s}$ is divisible by the defining section of D_Y. Let $\underline{s}'$ denote the residual section in $(f_Y)^*(\pi_Y^* \mathbf{F} \otimes \mathbf{H}) \otimes \mathcal{O}(-D_Y)$. Then by proposition 8 $f_Y^* \underline{s}$ is the image of $\underline{s}'$ under $(f_Y)^*(\pi_Y^* \mathbf{F} \otimes \mathbf{H}) \otimes \mathcal{O}(-D_Y) \mapsto (f_Y)^*(\pi_Y^* \mathbf{F} \otimes \mathbf{H})$.

Consider the closure of $s_{\mathbf{E}}(X) - G$ in $\tilde{Y}$, which is nothing but the strict (proper) transformation of $s_{\mathbf{E}}(X) \subset Y$ under the composite blowing ups. We denote the resulting scheme by R. Because $\underline{s}'$ is the residual section of $(f_Y)^* \underline{s}$ vanishing on $R \cup D_Y$, it is clear that the zero locus of $\underline{s}'$ in $\tilde{Y}$, $Z(\underline{s}')$, is equal to R, and is of codimension e in $\tilde{Y}$. The section $\underline{s}'$ may not be regular since R may not be always smooth. Nevertheless by example 14.3.1. on page 251 of [F], when $[Z(\underline{s}')] = \sum_i \mathbf{m}_i [\Omega_i]$, we know that $[\mathbf{Z}(\underline{s}')] = \sum_i \mathbf{e}_i [\Omega_i]$ with $\mathbf{e}_i \leq \mathbf{m}_i$. But its zero locus $R = Z(\underline{s}')$ is birational to $s_{\mathbf{E}}(X) \cong X$, the initial base space. Because X is reduced, so is R, then $[R] = m[R_{red}]$ with $m = 1$. Thus we may still conclude that $\mathbf{Z}(\underline{s}') = [R]$ without the regularity assumption [45] on R. Let i_R denote the inclusion $i_R : R \mapsto \tilde{Y}$. Then $i_{R*}[R] = i_{R*}[Z(\underline{s}')] = i_{R*} \mathbf{Z}(\underline{s}') = c_e(f_Y^*(\pi_Y^* \mathbf{F} \otimes \mathbf{H}) \otimes \mathcal{O}(-D_Y)) \cap [\tilde{Y}]$.

Thirdly the bundle map $\mathbf{E} \oplus \mathbf{C} \mapsto \mathbf{E}$ on X induces a tautological regular section s_{tauto} of $\pi_Y^* \mathbf{E} \otimes \mathbf{H}$ on Y vanishing at $s_{\mathbf{E}}(X) \subset Y$. The pull-back section $(f_Y)^* s_{tauto}$ of $(f_Y)^*(\pi_Y^* \mathbf{E} \otimes \mathbf{H})$ defines a zero locus $Z((f_Y)^* s_{tauto}) = (f_Y)^{-1}(Z(s_{tauto})) = (f_Y)^{-1}(s_{\mathbf{E}}(X))$. Because none of the sub-cones G_i we blow up is contained in $s_{\mathbf{E}}(X) = \mathbf{C}_{\rho_0}$, the sub-scheme $f_Y^{-1}(s_{\mathbf{E}}(X))$ can be identified with the closure of $s_{\mathbf{E}}(X) - G$ in $\tilde{Y}$, which is nothing but R. By the same reasoning as above, we have

$$i_{R*}[R] = i_{R*}[Z((f_Y)^* s_{tauto})] = i_{R*} \mathbf{Z}((f_Y)^* s_{tauto}) = c_e((f_Y)^*(\pi_Y^* \mathbf{E} \otimes \mathbf{H})) \cap [\tilde{Y}].$$

[44] We have skipped the dependence of $\tilde{Y}$ on the choices of the blowing ups.

[45] Nevertheless, R is still of the right codimension and is regular on a dense open subset.

Fourthly the section $s_0(X) \subset \mathbf{E} \subset \mathbf{P}(\mathbf{E} \oplus 1)$ can be viewed as a sub-scheme in Y, denoted by the same symbol. Because $\pi_{\mathbf{E}}|_{s_0(X)} : s_0(X) \mapsto X$ induces an isomorphism and $s_0(X) \cap \mathbf{P}(\mathbf{E}) = \emptyset$ ($\mathbf{P}(\mathbf{E})$ is at infinity), the hyperplane line bundle $\mathbf{H} \mapsto Y$ is trivialized over $s_0(X)$ by its cross section which vanishes exactly on $\mathbf{P}(\mathbf{E})$. So $\mathbf{H}|_{s_0(X)} \cong \mathbf{C}$, and $\pi_Y^* \mathbf{E} \otimes \mathbf{H}|_{s_0(X)} \cong \mathbf{E}$. Then under the Gysin homomorphism the s_0 pull-back of the formula $c_e(\pi_Y^* \mathbf{E} \otimes \mathbf{H}) \cap [Y] = [s_{\mathbf{E}}(X)] \in \mathcal{A}.(Y)$ by $s_0^* : \mathcal{A}.(Y) \mapsto \mathcal{A}._{-e}(X)$ becomes $c_e(\mathbf{E}) \cap [X] = s_0^*[X] \in \mathcal{A}.(X)$.

Next we construct an embedding of the blown up scheme $\tilde{X}$ into the blown up projective bundle $\tilde{Y}$.

One notices that the intersection $s_0(X) \cap \mathbf{P}(\mathbf{C}_{\rho_i} \oplus 1)$ lies inside the cone $\mathbf{C}_{\rho_i}$ and is equal to $s_0(X) \cap \mathbf{C}_{\rho_i}$. By corollary 7.15. on page 165 of [Ha] and the subsequent definition, the strict transform of a closed sub-scheme of a scheme theoretical blowing up along a given blowing up center can be identified to be the blowing up of this sub-scheme along its intersection with the given blowing up center. Thus one finds that the closure of $s_0(X) - \cup_{i \in I} \mathbf{C}_{\rho_i}$ inside $\tilde{Y}$ is isomorphic to $\tilde{X}$, the consecutive blowing ups of X along strict transforms of the forms $\pi_{\mathbf{E}}((\cup_{i \in I_l} \mathbf{C}_{\rho_i}) \cap s_0(X))$ with $l = 1, 2, \cdots, r$. By abusing the notation slightly, we fix such an identification and still denote the resulting sub-scheme of $\tilde{Y}$ by the same symbol $\tilde{X}$.

We have established the following crucial facts after identifying the strict transformation of $s_0(X)$ in $\tilde{Y}$ with $\tilde{X}$,

(i). $\mathbf{H}|_{\tilde{X}} \cong \mathbf{C}$.

(ii). $D_Y|_{\tilde{X}} = D$.

(iii). $(f_Y)^*(\pi_Y^* \mathbf{F} \otimes \mathbf{H}) \otimes \mathcal{O}(-D_Y)|_{\tilde{X}} = f^* \mathbf{F} \otimes \mathcal{O}(-D)$.
and

(iv). The sections $\underline{s}'$, s' of the vector bundles in (iii) are compatible. Namely, $\underline{s}'|_{\tilde{X}} = s'$.

Set $i_{\tilde{X}} : \tilde{X} \mapsto \tilde{Y}$ and set $i : Z(s') \mapsto \tilde{X}$ to be the inclusion maps.

Then we may conclude that for all $\alpha \in \mathcal{A}_e(\tilde{X})$ the following identification argument: By using (ii)., (iii)., the projection formula of Chern classes (see page 3.2.(c), page 50 of [F]) and the relationship between the global and the localized top Chern classes of $(f_Y)^*(\pi_Y^* \mathbf{F} \otimes \mathbf{H} \otimes \mathcal{O}(-D_Y))$,

$$i_{\tilde{X}*}\{c_e(f^* \mathbf{F} \otimes \mathcal{O}(-D)) \cap \alpha\} = i_{\tilde{X}*}(c_e(i_{\tilde{X}}^*(f_Y)^*(\pi_Y^* \mathbf{F} \otimes \mathbf{H} \otimes \mathcal{O}(-D_Y))) \cap \alpha)$$

$$= c_e((f_Y)^*(\pi_Y^* \mathbf{F} \otimes \mathbf{H} \otimes \mathcal{O}(-D_Y))) \cap [\tilde{Y}] \cap i_{\tilde{X}*}\alpha = i_{R*}\mathbf{Z}(\underline{s}') \cap i_{\tilde{X}*}\alpha.$$

And by the defining formula of the localized top Chern class of $\underline{s}'$ and the concluding equality of the second and the third statements on page 41,

$$= i_{R*}(\underline{s}')^![\tilde{Y}] \cap i_{\tilde{X}*}\alpha = i_{R*}[R] \cap i_{\tilde{X}*}\alpha = (c_e(f_Y^*(\pi_Y^* \mathbf{E} \otimes \mathbf{H})) \cap [\tilde{Y}]) \cap i_{\tilde{X}*}\alpha.$$

Then by projection formula of Chern classes again,

$$= c_e((f_Y)^*(\pi_Y^*\mathbf{E}\otimes\mathbf{H}))\cap i_{\tilde{X}*}\alpha = i_{\tilde{X}*}\{c_e(i_{\tilde{X}}^*(f_Y)^*(\pi_Y^*\mathbf{E}\otimes\mathbf{H}))\cap\alpha\}.$$

Then by (i). above we know $\mathbf{H}|_{\tilde{X}} = \mathbf{C}$, so finally the above expression

$$= i_{\tilde{X}*}\{c_e(f^*(\mathbf{E})\cap i_{\tilde{X}*}\alpha\}.$$

Because $\tilde{X}\mapsto pt$ factorizes as $\tilde{X}\xrightarrow{i_{\tilde{X}}}\tilde{Y}\mapsto pt$, this implies that

$$\int_{\tilde{X}} c_e(f^*\mathbf{F}\otimes\mathcal{O}(-D))\cap\alpha = \int_{\tilde{X}} c_e(f^*\mathbf{E})\cap i_{\tilde{X}*}\alpha\in\mathcal{A}_0(pt),$$

for all $\alpha\in\mathcal{A}_e(\tilde{X})$.

Therefore $c_e(f^*\mathbf{F}\otimes\mathcal{O}(-D))\cap[\tilde{X}]\overset{n}{=}c_e(f^*\mathbf{E})\cap[\tilde{X}]$. The proposition is proved. □

Remark 8 *In the proof of this proposition, if we have the knowledge of the injectivity* $0\mapsto\mathcal{A}_{m-e}(\tilde{X})\mapsto\mathcal{A}_{m-e}(\tilde{Y})$*, then our argument implies a stronger result that* $c_e(f^*\mathbf{F}\otimes\mathcal{O}(-D))\cap[\tilde{X}] = c_e(f^*\mathbf{E})\cap[\tilde{X}]\in\mathcal{A}_{m-e}(\tilde{X})$*. In our paper's main application to the algebraic family Seiberg-Witten invariants, the top Chern classes are paired with other cycle classes and then pushed-forward to* $\mathcal{A}_0(pt)$ *to form algebraic family Seiberg-Witten invariants. Our result ensures that one may replace* $c_e(f^*\mathbf{F}\otimes\mathcal{O}(-D))$ *by* $c_e(f^*\mathbf{E})$ *whenever* $c_e(f^*\mathbf{F}\otimes\mathcal{O}(-D))$ *appears in an integration of the top intersection pairing. For this purpose, one may view them as "equal" without causing potential confusion.*

There are many different blowing up sequences which can bring $\cup_{i>0}\mathbf{P}(\mathbf{C}_{\rho_i}\oplus 1)$ into an exceptional Cartier divisor. If one chooses to blow up the whole $\cup_{i>0}\mathbf{P}(\mathbf{C}_{\rho_i}\oplus 1)$ in $\mathbf{P}(\mathbf{E}\oplus\mathbf{C})$ all at once, the proposition implies that $c_e(\mathbf{F})\cap[X]-c_e(\mathbf{E})\cap[X]$ is equal to the push-forward of the local contribution of top Chern class from $Z=\cup_{i>0}supp(\mathbf{C}_{\rho_i})$ into X. If one chooses to group various irreducible $\mathbf{P}(\mathbf{C}_{\rho_i}\oplus 1)$ into different sub-schemes and blow up consecutively, one may apply proposition 8 and lemma 11 inductively and get a sum of cycles supported in $Z=supp(\cup_{i>0}\mathbf{C}_{\rho_i})$. It is natural to wonder if the result is invariant to the various choices of the orders of the blowing ups.

A corollary of proposition 9 is the following,

Corollary 2 *With the same notations as in proposition 9, the expression* $\sum_{1\leq i\leq e}(-1)^{i-1}c_{e-i}(\mathbf{F})\cap D^{i-1}\cap[D]$ *is numerically equivalent to* $\{c(\mathbf{F}|_{Z(s)\cap Z})\cap s(Z(s)\cap Z,X)\cap[X]\}_{m-e}$ *and is therefore independent to the choices of blowing up processes.*

Proof: Let $I=\coprod_{1\leq l\leq r}I_l$ be a partition of the index set I and let

$$\tilde{X}=\tilde{X}_r\mapsto\tilde{X}_{r-1}\mapsto\tilde{X}_{r-2}\mapsto\cdots\tilde{X}_1\mapsto\tilde{X}_0=X$$

be the sequence of blowing up processes where the $l-$th blowing up $\tilde{X}_l \mapsto \tilde{X}_{l-1}$ is centered at the strict transform of $\pi_{\mathbf{E}}(\mathbf{s}_0 \cap (\cup_{i\in I_l}\mathbf{C}_{\rho_i}))$ under $\tilde{X}_{l-1} \mapsto \tilde{X}_0$.

First we notice that the section s_0 does not intersect with the infinity of the projective bundle $\mathbf{P}(\mathbf{E}) \subset \mathbf{P}(\mathbf{E}\oplus 1)$ and therefore $s_0 \cap (\cup_{i\in I_l}\mathbf{C}_{\rho_i}) = s_0 \cap (\cup_{i\in I_l}\mathbf{P}(\mathbf{C}_{\rho_i}\oplus 1))$ for all $1 \leq l \leq r$.

Let us fix a few notations. Let $\tilde{h}_l : \tilde{X} \mapsto \tilde{X}_l$ be the blowing down map from the final ($r-$th blowing up) to the $l-$th intermediate blowing up of X. Let D_l, $1 \leq l \leq r$ denote the exceptional divisor of $\tilde{X}_l \mapsto \tilde{X}_{l-1}$ indexed by the subscript l. Let $\tilde{D}_l$ denote its pre-image $\tilde{h}_l^{-1}(D_l) \subset \tilde{X}$.

Then by an induction argument, proposition 8 implies the following identities on the Chern classes,

$$c_e(\tilde{h}_0^*\mathbf{F}\otimes\otimes_{j\leq l-1}\tilde{h}_j^*\mathcal{O}(-D_j))\cap[\tilde{X}_r] - c_e(\tilde{h}_0^*\mathbf{F}\otimes\otimes_{j\leq l}\tilde{h}_j^*\mathcal{O}(-D_j))\cap[\tilde{X}_r]$$
$$= i_{\tilde{D}_l *}\sum_{1\leq i\leq e}(-1)^{i-1}c_{e-i}(\tilde{h}_0^*\mathbf{F}\otimes\otimes_{j\leq l-1}\mathcal{O}(-\tilde{D}_j))\cap\tilde{D}_l^{i-1}\cap[\tilde{D}_l],$$

for $1 \leq l \leq r$. If we sum up all these r equations, a simple cancellation of the intermediate terms leads to the final equation

$$c_e(\tilde{h}_0^*\mathbf{F})\cap[\tilde{X}] - c_e(\tilde{h}_0^*\mathbf{F}\otimes\otimes_{1\leq l\leq r}\mathcal{O}(-\tilde{D}_l))\cap[\tilde{X}] = \sum_{1\leq l\leq r} i_{\tilde{D}_l *}\sum_{1\leq i\leq e}(-1)^{i-1}c_{e-i}(\tilde{h}_0^*\mathbf{F}\otimes\otimes_{j\leq l-1}\mathcal{O}(-\tilde{D}_j))\cap\tilde{D}_l^{i-1}\cap[\tilde{D}_l].$$

By realizing $\mathcal{O}(D) = \otimes_{1\leq l\leq r}\mathcal{O}(\tilde{D}_l)$, the left hand side of the identity is $c_e(\tilde{h}_0^*\mathbf{F})\cap[\tilde{X}] - c_e(\tilde{h}_0^*\mathbf{F}\otimes\mathcal{O}(-D))\cap[\tilde{X}]$. Thus the right hand side can be identified with $\sum_{1\leq i\leq e}(-1)^{i-1}c_{e-i}(f^*\mathbf{F})\cap D^{i-1}\cap[D]$ by the well known property on the top Chern class. On the other hand, proposition 9 has implied that the push-forward of the left hand side under $\tilde{X} \mapsto X$ is numerically equivalent to $c_e(\mathbf{F})\cap[X] - c_e(\mathbf{E})\cap[X]$ and therefore is independent to all the grouping and the ordering choices involved in the blowing ups of $\pi_{\mathbf{E}}(s_0 \cap (\cup_{i>0}\mathbf{C}_{\rho_i})) \subset X$. In particular, we may take $I = I_1, r = 1$ to be the single partition of I and the $f : \tilde{X} \mapsto X$ is constructed from X by a single blowing up centered at $\pi_{\mathbf{E}}(\mathbf{s}_0 \cap (\cup_{i>0}\mathbf{C}_{\rho_i}))$. In this case, the identification of $f_* i_{D*}\sum_{1\leq i\leq e}(-1)^{i-1}c_{e-i}(f^*\mathbf{F})\cap D^{i-1}\cap[D]$ with the local contribution of top Chern class $\{c(\mathbf{F}|_{Z(s)\cap Z})\cap s(Z(s)\cap Z, X)\cap[X]\}_{m-e}$ is the direct consequence of lemma 11. $\square$

Remark 9 *An alternative way to prove corollary 2 and show that it is independent to the ordering of the blowing ups is to notice that $f(D)$ is always equal to $Z(s)\cap Z$ no matter which blowing up sequence we choose. Then by realizing $\sum_{i>0}(-1)^{i-1}D^{i-1} = s(D,\tilde{X})$ and by using proposition 4.2.(a) of*[46] *[F],*

$$f_* s(D,\tilde{X}) = deg(\tilde{X}/X)s(f(D), f(\tilde{X})) = 1\cdot s(Z(s)\cap Z, X),$$

one may identify $f_*\big(c(f^*\mathbf{F})\cap(\sum_{i>0}(-1)^{i-1}D^{i-1}[D])\big)$ with $(c(\mathbf{F})\cap s(Z(s)\cap Z, X)$.

[46] Cited in proposition 17.

The observation in remark 9 will be used in proving proposition 16 in section 6.1.

4.1 Some Observation about Residual Intersection Formula of Top Chern Classes

In this subsection, we consider the following geometric setting. Let X be a purely m dimensional scheme and let $\mathbf{E}$ be a rank e vector bundle over X. Suppose that $\mathbf{E}_0$ is a rank e_0 sub-bundle of $\mathbf{E}$ with a section $s_0 : X \mapsto \mathbf{E}_0$, and suppose that we have the following bundle short exact sequence,

$$0 \mapsto \mathbf{E}_0 \mapsto \mathbf{E} \mapsto \mathbf{E}/\mathbf{E}_0 \mapsto 0.$$

The section s_0 and the bundle injection $\mathbf{E}_0 \mapsto \mathbf{E}$ induces a section $s : X \mapsto \mathbf{E}$ and we know $Z(s_0) = Z(s)$.

We raise the following question:

Question: How are the residual intersection formulae of the top Chern classes of $\mathbf{E}_0$ and $\mathbf{E}$ related to each other?

More precisely, let $Z_1, Z_2, \cdots, Z_k$ be a finite number of closed proper subschemes of X. One may blow up the strict transforms (under the previous blowing ups) of $Z(s_0) \cap Z_i$, $1 \leq i \leq k$ consecutively and get a residual intersection formulae of top Chern classes of $\mathbf{E}_0$. On the other hand, we may blow up the strict transforms of $Z(s) \cap Z_i$, $1 \leq i \leq k$ consecutively and get the residual top Chern classes of $\mathbf{E}$. Because $Z(s_0) = Z(s)$, we expect these two residual intersection formulae to be closed related. This is the content of the following proposition,

Proposition 10 *Let $\tilde{X}$ denote the scheme repeatedly blown up from X centered at the strict transforms of $Z(s_0) \cap Z_i$, $1 \leq i \leq k$ and let $f : \tilde{X} \mapsto X$ denote the composite blow down projection map. Let $D = f^{-1}(\cup_i Z_i \cap Z(s_0))$ be the exceptional Cartier divisor above $\cup_i Z_i \cap Z(s_0) = \cup_i Z_i \cap Z(s)$. Let $(f^* s_0)'$ and $(f^* s)'$ denote the residual sections in $f^*\mathbf{E}_0 \otimes \mathcal{O}(-D)$ and $f^*\mathbf{E} \otimes \mathcal{O}(-D)$ of $f^* s_0 \in \Gamma(\tilde{X}, f^*\mathbf{E}_0)$ and $f^* s \in \Gamma(\tilde{X}, f^*\mathbf{E})$, respectively.*

By proposition 8 there is a residual intersection formula of the localized top Chern class of $f^*\mathbf{E}_0$,

$$\mathbf{Z}(f^* s_0) = \mathbf{Z}((f^* s_0)') + \sum_{1 \leq i \leq e_0} (-1)^{i-1} c_{e_0 - i}(\mathbf{E}_0|_D) \cap D^{i-1} \cap [D].$$

Suppose we cap the above formula with the top Chern class $c_{e-e_0}(f^*(\mathbf{E}/\mathbf{E}_0)|_D)$, one gets the corresponding residual intersection formula of the localized top Chern class of $f^* s$,

$$\mathbf{Z}(f^* s) = \mathbf{Z}((f^* s)') + \sum_{1 \leq i \leq e} (-1)^{i-1} c_{e-i}(\mathbf{E}|_D) \cap D^{i-1} \cap [D].$$

Proof of proposition 10: Recall that (see [F] page 244 and proposition 6.1.(a) page 94), $\mathbf{Z}(f^*s_0)$ is equal to $\{c(f^*\mathbf{E}_0)\cap s(Z(f^*s_0),\tilde{X})\}_{m-e_0}$ (we have skipped the bundle restriction notation of $f^*\mathbf{E}_0$ to $Z(f^*s_0)$ to simplify the notation). By proposition 13 of [Liu5], $\{c(f^*\mathbf{E}_0)\cap s(Z(f^*s_0),\tilde{X})\}_{m-e_0+r}=0$ for all negative $r\in -\mathbf{N}$, i.e. the localized contribution of top Chern class is the lowest degree term of the cycle class formed by the total Chern/Segre classes. Because $Z(f^*s_0)=Z(f^*s)$, we may compare it with[47]

$$\{c(f^*\mathbf{E})\cap s(Z(f^*s),\tilde{X})\}_{m-e}=\{c(f^*\mathbf{E}_0)\cap c(f^*(\mathbf{E}/\mathbf{E}_0))\cap s(Z(f^*s),\tilde{X})\}_{m-e}=\{c(f^*\mathbf{E}_0)\cap c(f^*(\mathbf{E}/\mathbf{E}_0))\cap s(Z(f^*s_0),\tilde{X})\}_{m-e}$$

$$=\sum_{r\geq 0}\{c(f^*\mathbf{E}_0)\cap s(Z(f^*s_0),\tilde{X})\}_{m-e_0-(e-e_0)+r}\cap c_r(f^*(\mathbf{E}/\mathbf{E}_0))=\sum_{r'\geq -(e-e_0)}\{c(f^*\mathbf{E}_0)\cap s(Z(f^*s_0),\tilde{X})\}_{m-e_0+r'}\cap c_{r'+(e-e_0)}(f^*\mathbf{E}/\mathbf{E}_0).$$

Because $\mathbf{E}/\mathbf{E}_0$ is of rank $e-e_0$, $c_{r'+(e-e_0)}(f^*(\mathbf{E}/\mathbf{E}_0))=0$ for all $r'\in\mathbf{N}$. So by the vanishing of terms with grades $m-e_0+r'$ for $r'<0$ in $\{c(f^*\mathbf{E}_0)\cap s(Z(f^*s_0),\tilde{X})\}$ the above sum is reduced to a single term and the result is nothing but the cap product of $\{c(f^*\mathbf{E}_0)\cap s(Z(f^*s_0),\tilde{X})\}_{m-e_0}$ with $c_{e-e_0}(f^*\mathbf{E}/\mathbf{E}_0)$.

The same discussion can be applied to $\mathbf{Z}((f^*s_0)')$ and $\mathbf{Z}((f^*s)')$ as well. Then the correspondence of the two formulae follows as the correspondence under capping with $c_{e-e_0}(\mathbf{E}/\mathbf{E}_0)$ has been shown to hold for two out of the three terms. □

Remark 10 *When there is only one $Z=Z_1$ and $\tilde{X}$ is the blown up of X along $Z_1\cap Z(s)$, the direct proof of the equality of $c_{e-e_0}(f^*(\mathbf{E}/\mathbf{E}_0)|_D)\cap\sum_{1\leq i\leq e_0}(-1)^{i-1}c_{e_0-i}(\mathbf{E}_0|_D)\cap D^{i-1}\cap[D]$ and $\sum_{1\leq i\leq e}(-1)^{i-1}c_{e-i}(\mathbf{E}|_D)\cap D^{i-1}\cap[D]$ follows from the identification with the localized top Chern class contribution of $Z\cap Z(s)$ as was done in lemma 11. When more than one Z_i is present, one may adopt the similar argument in the proof of corollary 2 and lemma 11 to identify them directly. As there is no new idea involved, we leave the details to the readers.*

5 Residual Intersection Formula and Inductive Blowing Ups of $X=\mathbf{P}(\mathbf{V}_{canon})$

We follow the same notations as in [Liu1], [Liu3] and [Liu5]. Let $\mathbf{M}(E)E=\sum_{i\leq n}m_iE_i$ be the sum of the exceptional divisors E_i with multiplicities m_i, $1\leq i\leq n$. To simplify our discussion, we require that $m_i\leq m_j$ for all $i<j$.

As in [Liu3], we take $(\Phi_{\mathbf{V}_{canon}\mathbf{W}_{canon}},\mathbf{V}_{canon},\mathbf{W}_{canon})$ over $M_n\times T(M)$ to be the canonical algebraic Kuranishi model of the class $C-\mathbf{M}(E)E$ with respect to $f_n: M_{n+1}\mapsto M_n$. We take the initial total space of our discussion

[47] We have changed r to $r'=r-(e-e_0)$ in the second line of the equalities.

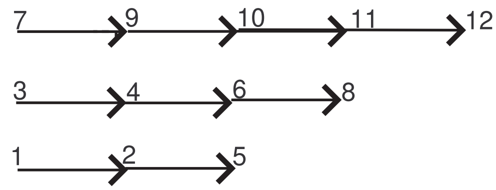

fig.4
a chain-like admissible graph with three connected components and 12 vertexes

to be $X = \mathbf{P}_{M_n \times T(M)}(\mathbf{V}_{canon})$. This space parametrizes all the curves in the non-linear system of C along the family $M_n \times T(M)$.

Recall from section 2 that the universal space M_n allows a stratification by the admissible strata Y_Γ, $\Gamma \in adm(n)$. We consider the finite set of admissible strata satisfying the **Special Condition** first stated in [Liu5] section 6.1:

For all the type I exceptional classes $e_1, e_2, \cdots, e_n$ of Y_Γ,

$\diamondsuit$ either

(i). $(C - \mathbf{M}(E)E) \cdot e_i < 0$, i.e. $\mathbf{M}(E)E \cdot e_i > 0$.

$\diamondsuit$ or

(ii). the condition $e_i^2 = -1$ holds, i.e. e_i is a type I -1 class.

This maximality special condition means there is no “redundant” type I classes which pair non-negatively with $C - \mathbf{M}(E)E$.

Recall from section 2 that the notation $adm(n)$ denotes the finite set of all n-vertex admissible graphs Γ. We introduce a few subsets of $adm(n)$ here.

Definition 6 *Let $\Delta(n) \subset adm(n)$ denote the subset of $adm(n)$ consisting of all n-vertex admissible graphs satisfying the* **special condition** *above. Let $adm_2(n) \subset adm(n)$ denote the subset of n-vertex admissible graphs satisfying the condition that each vertex has at most one direct descendent.*

The graphs [48] in $adm_2(n)$ may have more than one connected component. Each component looks like a chain of vertexes connected by a chain of arrows. We will refer to them as chain-like admissible graphs in the following discussion.

Firstly we point out that the union of the closure of all such strata Y_Γ, $\Gamma \in \Delta(n)$, form a large closed subset of M_n.

Proposition 11 *The closed subset $\cup_{\Gamma \in \Delta(n) - \{\gamma_n\}} Y(\Gamma)$ is of at least complex codimension two in M_n. Its complement can be expressed as $\coprod_{\Gamma \in P} Y_\Gamma$ for some $P \subset adm_2(n)$.*

Proof: Firstly, we know that $\Delta(n) \neq \emptyset$ because the admissible graph γ_n with no one-edges is in $\Delta(n)$.

[48] See fig.4 for an example.

We know that each $Y(\Gamma)$ is closed, so $\cup_{\Gamma\in\Delta(n),\Gamma\neq\gamma_n}Y(\Gamma)$ is a finite union of closed sets and is closed.

We observe that if for some $i\leq n$, the i-th vertex of Γ contains more than one direct descendent vertex, then the intersection pairing of $e_i=E_i-\sum_{j_i}E_{j_i}$ and $C-\mathbf{M}(E)E$ is negative because the assumption $m_i\leq m_j$, for $i\leq j$,

$$e_i\cdot(C-\mathbf{M}(E)E)=(E_i-\sum_{j_i}E_{j_i})\cdot(C-\sum_k m_kE_k)=m_i-\sum_{j_i}m_{j_i}<0.$$

If the index i has exact one direct descendent in $\Gamma\in adm(n)$, denoted as j, then the intersection pairing of this -2 class $e_i=E_i-E_j$ and $C-\mathbf{M}(E)E$ is $m_i-m_j\leq 0$.

Take an arbitrary $\Gamma\notin adm_2(n)$, then there must be some index $i\leq n$ such that the $i-$th vertex contains more than one direct descendent in Γ. Given the graph Γ, one may construct the type I exceptional classes associated to it following the recipe in section 2: Given any index $i, 1\leq i\leq n$, let j_i be the indexes of all the direct descendents of i in the graph Γ. Then take $e_i=E_i-\sum_{j_i}E_{j_i}$.

Among all such $e_i, 1\leq i\leq n$, consider all the type I classes $e_{k_i}, 1\leq i\leq p$ (for some p depending on both Γ and $\mathbf{M}(E)E$), attached to Γ which have negative pairings with the given $C-\mathbf{M}(E)E$. Each e_{k_i} is represented by an effective fiberwise divisor in the fiber algebraic surfaces of $M_{n+1}\mapsto M_n$ over a smooth locus $Y(\Gamma_{e_{k_i}})$ with a complex codimension[49] equal to the number of direct descendents of k_i in Γ. This locus is usually called the existence locus of e_{k_i}.

By proposition 3 in section 2 one can construct an admissible graph Γ_0 from e_{k_i}, $1\leq i\leq p$, satisfying $e_{k_i}\cdot e_{k_j}\geq 0$ for $i\neq j$. Then by the corollary of proposition 4 and remark 5 we have $Y(\Gamma_0)=\cap_{1\leq i\leq p}Y(\Gamma_{e_{k_i}})$ and $Y(\Gamma_0)$ is the locus in M_n over which $e_{k_1},e_{k_2},\cdots,e_{k_p}$ co-exist as effective curve/divisor classes.

Because these e_{k_i} are effective over Y_Γ, so we know that Γ is a degeneration of Γ_0, i.e. $\Gamma<\Gamma_0$, and $Y_\Gamma\subset Y(\Gamma_0)$. By the construction of Γ_0, all these $e_{k_i}, 1\leq i\leq p$ are the only type I exceptional classes associated to Γ_0 which are not -1 classes. Therefore it satisfies the **special condition** and $\Gamma_0\subset\Delta(n)$.

Thus we conclude that every stratum Y_Γ, $\Gamma\notin adm_2(n)$, must be contained in $\cup_{\Gamma\in\Delta(n)-\{\gamma_n\}}Y(\Gamma)$.

Secondly, we separate into two cases to determine the dimension of $\cup_{\Gamma\in\Delta(n)-\{\gamma_n\}}Y(\Gamma)$. If all the singular multiplicities are equal $m_1=m_2=\cdots=m_n$, then $C-\mathbf{M}(E)E$ has vanishing pairings with all the -2 type I exceptional classes of the general form E_i-E_j. The admissible graphs in $adm_2(n)$ are exactly those graphs whose associated type I exceptional classes are either -1 or -2 classes. In this case, the codimension of $\cup_{\Gamma\in\Delta(n)-\{\gamma_n\}}Y(\Gamma)$ is exactly two because all the codimension one admissible strata are parametrized by some very simple graphs [50] in $adm_2(n)$. The index set P can be taken to be the whole $adm_2(n)$.

[49] $=codim_{\mathbf{C}}\Gamma$.

[50] Indeed, graphs with a single one-edge.

If there exists a pair of singular multiplicities $m_i < m_j$ for $1 \leq i < j \leq n$, then $C - \mathbf{M}(E)E \cdot (E_i - E_j) < 0$. Consider the admissible graph $\Gamma_{i,j}$ with a unique one-edge starting at the $i-$th vertex and ending at the $j-$th vertex. Apparently it belongs to $adm_2(n)$. Because $m_i < m_j$, this graph $\Gamma_{i,j}$ also belongs to $\Delta(n)$. In this case, codimension of $\cup_{\Gamma \in \Delta(n) - \{\gamma_n\}} Y(\Gamma)$ is one. The index subset P is then chosen to be the proper subset of $adm_2(n)$, removing all the chain-like admissible graphs of the type $\Gamma_{i,j}$ with single edges from i to j, for all the pairs $m_i < m_j$, $i < j$. $\square$

In the paper [Liu4], we have considered the set Q and describe a curve-counting scheme based on Q. In our current setting of type I exceptional classes, define Q to be the finite set of all the classes $E_i - \sum_{i<j} E_j$ which pair negatively with $C - \mathbf{M}(E)E$. The Set Q encodes all the possible type I exceptional classes which can appear above the family $M_{n+1} \mapsto M_n$.

Recall the definition of the type I exceptional cone over a point $b \in M_n$,

Definition 7 *Let b be an arbitrary point in M_n, define the type I exceptional cone over b, $\mathcal{EC}_b(C - \mathbf{M}(E)E; Q)$, to be the real cone in $H^{1,1}(M_{n+1}|_b, \mathbf{R})$ generated by all the type I exceptional classes effective over b which pair with $C - \mathbf{M}(E)E$ negatively.*

According to proposition 4 of [Liu4], the cone is always simplicial, and we call the primitive generators at the 1-edge the extremal generators of the cone.

Because the fiber bundle $M_{n+1} \mapsto M_n$ has no non-trivial monodromy, one can discuss about the change of the cone un-ambiguously. The variation of $\mathcal{EC}_b(C - \mathbf{M}(E)E; Q)$ with respect to b gives us important information about how to organize the admissible strata $Y_\Gamma, \Gamma \in \Delta(n)$.

Given a $Y(\Gamma)$, $\Gamma \in \Delta(n)$, the type I exceptional cone $\mathcal{EC}_b(C - \mathbf{M}(E)E; Q)$ may vary when b specializes to the boundary points $\coprod_{\Gamma' < \Gamma} Y_{\Gamma'}$.

Let $\mathcal{C}_\Gamma$ denote the type I exceptional cone $\mathcal{EC}_b(C - \mathbf{M}(E)E; Q)$ constant for all $b \in Y_\Gamma$. There is a distinguished locally closed subset $S_\Gamma \subset Y(\Gamma)$, S_Γ (should be denoted by $S_{\mathcal{C}_\Gamma}$ if we follow the notation in [Liu4]) over which the exceptional cone $\mathcal{EC}_b(C - \mathbf{M}(E)E; Q) \equiv \mathcal{C}_\Gamma$ remain unchanged.

Because for all $b \in Y_{\Gamma'}$, their $\mathcal{EC}_b(C - \mathbf{M}(E)E; Q)$ remain constant [51], S_Γ itself is a union of admissible strata and one may write S_Γ formally as $Y_\Gamma \coprod_{Y_{\Gamma'} \cap S_\Gamma \neq \emptyset} Y_{\Gamma'}$.

Lemma 13 *The union $\cup_{\Gamma \in \Delta(n)} Y(\Gamma) = Y_{\gamma_n} \cup_{\Gamma \in \Delta(n) - \{\gamma_n\}} Y(\Gamma) \subset M_n$ is equal to the disjoint union $\coprod_{\Gamma \in \Delta(n)} S_\Gamma$.*

Proof: Following the proof of proposition 11, for all Y_Γ over which at least one type I exceptional class pairs negatively with $C - \mathbf{M}(E)E$, $Y_\Gamma \subset S_{\Gamma_0}$ for some unique Γ_0 constructed (in the proof of proposition 11 through the usage of proposition 3) by the co-existence of different type I curves.

[51] Which may be different from $\mathcal{C}_\Gamma$ though.

On the other hand, an effective type I exceptional class pairing negatively with $C-\mathbf{M}(E)E$ above Y_Γ still remains effective and pairs negatively with $C-\mathbf{M}(E)E$ over the boundary $\partial Y_\Gamma = Y(\Gamma) - Y_\Gamma$. But it may break into more than one irreducible component. This implies that over any such degenerated stratum $Y_{\Gamma'} \subset Y(\Gamma)$ there must still exist at least one type I class pairing negatively with $C-\mathbf{M}(E)E$.

This shows that $\cup_{\Gamma\subset\Delta(n)} Y(\Gamma) \subset \cup_{\Gamma\in\Delta(n)} S_\Gamma$. The opposite inclusion follows from the inclusion $S_\Gamma \subset Y(\Gamma)$ for $\Gamma \in \Delta(n)$. Finally $S_\Gamma \cap S_{\Gamma'} = \emptyset$ if $\Gamma \neq \Gamma'$ in $\Delta(n)$. It is because $\mathcal{C}_\Gamma \neq \mathcal{C}_{\Gamma'}$ and by definition of S_Γ, they can not overlap. So the union is a disjoint union. $\Box$

Among all such S_Γ, $\Gamma \in \Delta(n)$, one may introduce a partial ordering $\succ$, as has been done in [Liu5] for a slightly more general setting. The partial ordering induces a partial ordering on the corresponding graphs in $\Delta(n)$, denoted by the same symbol.

Definition 8 *Let $\Gamma_1, \Gamma_2 \in \Delta(n)$. The graph Γ_1 is said to be greater than Γ_2 under the partial ordering $\succ$, denoted as $\Gamma_1 \succ \Gamma_2$, if $\mathcal{C}_{\Gamma_1} \subset \mathcal{C}_{\Gamma_2}$.*

Please refer to page 69, fig.7 for an example. In that example, the smaller cone is generated by $E_1 - E_2 - E_3 - E_7$, E_2, E_3, E_4, E_5, E_6, E_7. The larger cone is generated by $E_1 - E_2 - E_3 - E_4 - E_7$, $E_2 - E_5 - E_6$, E_3, E_4, E_5, E_6, E_7.

A sufficient condition to check whether $\Gamma_1 \succ \Gamma_2$ is the following,

Lemma 14 *Suppose that $\overline{S_{\Gamma_1}} = Y(\Gamma_1)$ intersects with S_{Γ_2} non-trivially, then $\Gamma_1 \succ \Gamma_2$.*

Proof: The follows from the fact that the cones get larger under degenerations of points from $b \in S_{\Gamma_1}$ to $b \in S_{\Gamma_2}$. $\Box$

Our goal is to study the local contribution of the algebraic family Seiberg-Witten invariant over $\cup_{\Gamma\in\Delta(n)} Y(\Gamma)$ and decompose the algebraic family Seiberg-Witten invariant $\mathcal{AFSW}_{M_{n+1}\times T(M)\mapsto M_n\times T(M)}(1, C-\mathbf{M}(E)E)$ or the restricted version $\mathcal{AFSW}_{M_{n+1}\times\{t_L\}\mapsto M_n\times\{t_L\}}(1, C-\mathbf{M}(E)E)$, for some $t_L \in T(M)$, into the various excess local contributions from $\cup_{\Gamma\in\Delta(n)-\{\gamma_n\}} Y(\Gamma)$ and the residual contribution from $M_n - \cup_{\Gamma\in\Delta(n)-\{\gamma_n\}} Y(\Gamma)$ based on the following two simple but fundamental observations,

Observation 1: The family algebraic Seiberg-Witten invariant is defined by the push-forward into $\mathcal{A}_0(pt)$ of the cap product of a certain power of $c_1(\mathbf{H})$ (determined by the dimension formula) with the top Chern class of the canonical obstruction bundle $c_{top}(\mathbf{H} \otimes \pi^*_{\mathbf{P}(\mathbf{V}_{canon})} \mathbf{W}_{canon})$.

Observation 2: The residual intersection formula of top Chern class allows us to decompose the total invariant contribution into the local contribution to some closed subset of $X = \mathbf{P}_{M_n\times T(M)}(\mathbf{V}_{canon})$ and the residual contribution. The residual contribution again involves the top Chern class of a modified bundle over a blown up space.

A direct but probably naive approach is to set $Z = \pi_X^{-1}(\cup_{\Gamma\in\Delta(n)-\{\gamma_n\}}Y(\Gamma)) \subset X$, the pre-image in the $\mathbf{P}(\mathbf{V}_{canon})$ of $\cup_{\Gamma\in\Delta(n)-\{\gamma_n\}}Y(\Gamma)$, and then apply the residual intersection formula (proposition 8 and lemma 11) to the vector bundle $E = \mathbf{H}\otimes\pi_X^*\mathbf{W}_{canon}$, the section $s = s_{canon}$ and $Z\cap Z(s_{canon})$. The apparent drawback of this approach is that the local contribution of the set $Z(s_{canon})\cap Z$ to the family invariant is very hard to enumerative directly, due to the complicated geometric structure of $\cup_{\Gamma\in\Delta(n)-\{\gamma_n\}}Y(\Gamma)$, Z and therefore $Z(s_{canon})\cap Z$.

Instead we construct a more refined consecutive blowing ups of sub-schemes in X and make use of $Y(\Gamma)$ as the co-existence locus of all the type I exceptional classes e_i, $1\leq i\leq n$ effectively. In section 6.1, theorem 4 of [Liu5] we have demonstrated that (under some additional special assumption [52]) the local contribution of the family invariant to $X\times_{M_n}Y(\Gamma)$ can be identified with the mixed family invariant of $C-\mathbf{M}(E)E-\sum_{1\leq i\leq p}e_{k_i}$ over $Y(\Gamma)$. This motivates us to consider the following refined approach in section 5.1.

5.1 The Repeated Blowing Ups of Sub-Loci in X

The element γ_n is apparently the largest element under $\succ$ in $\Delta(n)$. Over the open top stratum the type I exceptional classes e_i are the -1 classes $E_i, 1\leq i\leq n$ and the exceptional cone [53] $\mathcal{C}_{\gamma_n}$ it generates is the smallest.

List all the $\Gamma\in\Delta(n)-\{\gamma_n\}$ and they form a finite graph (each $\Gamma\in\Delta(n)-\{\gamma_n\}$ being a vertex in the graph) under the partial ordering $\succ$. For all $\Gamma\in\Delta(n)-\{\gamma_n\}$, we consider the fiber product $X\times_{M_n}Y(\Gamma)$.

By definition the family moduli space $\mathcal{M}_{C-\mathbf{M}(E)E}$ over $M_n\times T(M)$ of curves dual to $C-\mathbf{M}(E)E$ collects all the curves within the fibers of $M_{n+1}\times T(M)\mapsto M_n\times T(M)$ which are dual to $C-\mathbf{M}(E)E$. When we use the canonical algebraic Kuranishi model, $Z(s_{canon}) = \mathcal{M}_{C-\mathbf{M}(E)E}$ for $s_{canon}\in\Gamma(X,\mathbf{H}\otimes\pi_X^*\mathbf{W}_{canon})$ (for the definitions of s_{canon}, $\mathbf{W}_{canon}$, please consult section 5.1 of [Liu3] and section 5, proposition 9, 10 of [Liu5]). So $\mathcal{M}_{C-\mathbf{M}(E)E}$ can be viewed as a sub-scheme of X and the inclusion $Z(s_{canon})\subset X$ induces the natural projection map to M_n. The schemes $Z(s_{canon})\cap(X\times_{M_n}Y(\Gamma)) = Z(s_{canon})\times_{M_n}Y(\Gamma)$, $\Gamma\in\Delta(n)-\{\gamma_n\}$, are sub-schemes of $Z(s_{canon})$ and the ultimate goal is to enumerate the residual contribution of ($t_L\in T(M)$)

$$\mathcal{AFSW}_{M_{n+1}\times\{t_L\}\mapsto M_n\times\{t_L\}}(1,C-\mathbf{M}(E)E) = c_1(\mathbf{H})^{p_g+rank_{\mathbf{C}}\mathbf{V}_{canon}-rank_{\mathbf{C}}\mathbf{W}_{canon}}\cap c_{top}(\mathbf{H}\otimes\pi_X^*\mathbf{W}_{canon}),$$

or

$$\mathcal{AFSW}_{M_{n+1}\times T(M)\mapsto M_n\times T(M)}(1,C-\mathbf{M}(E)E) = c_1(\mathbf{H})^{p_g+rank_{\mathbf{C}}\mathbf{V}_{canon}-rank_{\mathbf{C}}\mathbf{W}_{canon}}\cap c_{top}(\mathbf{H}\otimes\pi_X^*\mathbf{W}_{canon}),$$

[52] See theorem 4 of [Liu5] for details.

[53] Generated by E_1, E_2, E_3, $\cdots$, E_n.

outside $\cup_{\Gamma\in\Delta(n)-\{\gamma_n\}} Z(s_{canon}) \times_{M_n} Y(\Gamma)$ and show that under some additional assumption on t_L, the residual contribution localizes to lie above the open sub-space $X \times_{M_n} Y_{\gamma_n}$.

We achieve this by blowing up inductively along the various loci $Z(s_{canon})\times_{M_n} Y(\Gamma)$ (or more precisely their strict transforms under the previous blowing ups), starting from the minimal Γ under $\succ$ and running in the reversed orders of $\succ$. We will discuss extensively regarding the ambiguities involved in the orders of the blowing ups and how "doesn't" it affect the enumeration process.

Suppose that $Y(\Gamma_2) \subset Y(\Gamma_1)$, $\Gamma_1, \Gamma_2 \in \Delta(n)$, and then $Z(s_{canon}) \times_{M_n} Y(\Gamma_2) \subset Z(s_{canon}) \times_{M_n} Y(\Gamma_1)$, $\mathcal{C}_{\Gamma_2} \supset \mathcal{C}_{\Gamma_1}$. If $\mathcal{C}_{\Gamma_1}$ is a proper sub-cone of $\mathcal{C}_{\Gamma_2}$, then $Y(\Gamma_2) = \overline{S_{\Gamma_2}}$ can never intersect S_{Γ_1} non-trivially. Otherwise at any of the intersection points the type I exceptional cone is $\mathcal{C}_{\Gamma_1}$, by the definition of S_{Γ_1}. But this intersection point is also in Y_{Γ_2}, or can by degenerated from points in Y_{Γ_2}, i.e. it is in $Y(\Gamma_2) - Y_{\Gamma_2}$. Thus $\mathcal{C}_{\Gamma_2} \subset \mathcal{C}_{\Gamma_1}$ (by degenerations of cones) and is impossible by our assumption $\Gamma_1 > \Gamma_2$, or equivalently $Y(\Gamma_1) \supset Y(\Gamma_2)$.

Therefore in such a situation the graph Γ_2 can never $\succ \Gamma_1$. In fact lemma 14 and $\overline{S_{\Gamma_1}} \cap S_{\Gamma_2} = Y(\Gamma_1) \cap S_{\Gamma_2} = S_{\Gamma_2} \neq \emptyset$ implies $\Gamma_1 \succ \Gamma_2$.

On the other hand, we have the following lemma regarding the possible relationship between two admissible graphs in $\Delta(n)$,

Lemma 15 *Let $\Gamma_1, \Gamma_2 \in \Delta(n)$ be two distinct admissible graphs satisfying the special maximality conditions (on page 47). If $Y(\Gamma_1) \cap Y(\Gamma_2) \neq \emptyset$, there are three mutually exclusive possibilities.*

(a). $\Gamma_1 \succ \Gamma_2$.

(b). $\Gamma_2 \succ \Gamma_1$.

(c). Neither $\Gamma_1 \succ \Gamma_2$ nor $\Gamma_2 \succ \Gamma_1$. But there exists a "refined" $\Gamma_3 \in \Delta(n)$ such that $\Gamma_i \succ \Gamma_3$ for both $i = 1, 2$. I.e. Γ_3 is smaller than Γ_1, Γ_2 simultaneously.

Proof of lemma 15: This can be shown by contradiction easily. Assuming that neither (a). nor (b). holds, then lemma 14 implies that both $S_{\Gamma_1} \cap Y(\Gamma_2) = S_{\Gamma_2} \cap Y(\Gamma_1) = \emptyset$. Along with the fact that $S_{\Gamma_1} \cap S_{\Gamma_2} = \emptyset$ for $\mathcal{C}_{\Gamma_1} \neq \mathcal{C}_{\Gamma_2}$, it implies that

$$Y(\Gamma_1) \cap Y(\Gamma_2) \subset (Y(\Gamma_1) - S_{\Gamma_1}) \cap (Y(\Gamma_2) - S_{\Gamma_2}).$$

Let $b \in Y(\Gamma_1) \cap Y(\Gamma_2)$. Because $b \notin S_{\Gamma_1} \cup S_{\Gamma_2}$ but $b \in \overline{S_{\Gamma_1}} \cap \overline{S_{\Gamma_2}}$, $\mathcal{EC}_b(C - \mathbf{M}(E)E; Q)$ contains both $\mathcal{C}_{\Gamma_1}, \mathcal{C}_{\Gamma_2}$ as proper sub-cones. Let e_{k_i}, $1 \leq i \leq p$ be the primitive type I generators of the simplicial cone $\mathcal{EC}_b(C - \mathbf{M}(E)E; Q)$. Because e_{k_i} are represented by irreducible curves $e_{k_i} \cdot e_{k_j} \geq 0$ for $i \neq j$, then by proposition 3, one can construct a $\Gamma_3 \in adm(n)$ associated with these e_{k_i}, $1 \leq i \leq p$. By proposition 4 the co-existence of these type I exceptional curves dual to $e_{k_i}, 1 \leq i \leq p$, characterizes the admissible stratum $Y_{\Gamma_3} \subset Y(\Gamma_3)$ over which e_{k_i}, $1 \leq i \leq p$, are represented as smooth type I exceptional curves in the fibers of $M_{n+1} \times_{M_n} Y_{\Gamma_3} \mapsto Y_{\Gamma_3}$ and all $e_j, j \notin \{k_1, k_2, \cdots, k_p\}$ are -1 classes. By definition of $\mathcal{EC}_b(C - \mathbf{M}(E)E, Q)$, $e_{k_i} \cdot (C - \mathbf{M}(E)E) < 0$, $1 \leq i \leq p$ and so $\Gamma_3 \in \Delta(n)$.

By our construction of Γ_3, we have $b \in S_{\Gamma_3}$ since $\mathcal{C}_{\Gamma_3} = \mathcal{E}\mathcal{C}_b(C - \mathbf{M}(E)E; Q)$. It is apparent that $b \in S_{\Gamma_3} \cap Y(\Gamma_i) = S_{\Gamma_3} \cap \overline{S_{\Gamma_i}}$, for $i = 1, 2$. Thus by lemma 14, $\Gamma_1 \succ \Gamma_3$, $\Gamma_2 \succ \Gamma_3$ simultaneously. $\Box$

Remark 11 *Because the ambiguity of choices of the point b, the graph Γ_3 constructed in the proof may not be unique.*

To apply the residual intersection theory to $Z(s_{canon}) \times_{M_n} Y(\Gamma)$, $\Gamma \in \Delta(n) - \{\gamma_n\}$ inductively, each $Z(s_{canon}) \times_{M_n} Y(\Gamma) \subset X = \mathbf{P}(\mathbf{V}_{canon})$ determines a blowup center and in the following we decide the order of blowing ups by upgrading the partial ordering $\succ$ into a linear ordering called $\models$.

$\diamondsuit$ Definition of $\models$:

Initially define the **current index set** to be $\Delta(n)$. Because $(\Delta(n), \succ)$ is a partial ordered finite set, there must be some (maybe non-unique) minimal elements in $\Delta(n)$ which are not larger than other element in $\Delta(n)$ under $\succ$.

(1). List all the minimal elements in the **current index set** $\Delta(n)$ under the partial ordering $\succ$. Select one of them (this introduces some ambiguity if the minimal elements are not unique[54]).

(2). Remove the selected element from the **current index set** and list all the minimal elements from the residual set. Define the new **current index set** to be the residual set. Select one of the minimal elements again.

(3). Go back to step (2)., then repeat the above process and iterate.

(4). After a finite number of times, one will exhaust the whole $\Delta(n)$ and determine a sequence of blowing up centers.

In this way we have determined a linear ordering on $\Delta(n)$, denoted by $\models$.

The discussion right in front of lemma 15 indicates the following: Suppose that we blow up the strict transforms of $Z(s_{canon}) \times_{M_n} Y(\Gamma)$ following the reversed ordering of $\models$ starting from the smallest element in $\Delta(n) - \{\gamma_n\}$. After blowing up the strict transform of $Z(s_{canon}) \times_{M_n} Y(\Gamma)$, in our set up we will never blow up any sub-locus completely lying inside $Z(s_{canon}) \times_{M_n} Y(\Gamma)$. In fact, the strict transformation of any such sub-locus will be blown up prior to the blowing up of the strict transformation of $Z(s_{canon}) \times_{M_n} Y(\Gamma)$, due to the fact they are smaller under the partial ordering $\succ$, and therefore the linear ordering $\models$.

Let us blow up X inductively along the strict transforms of the various $Z(s_{canon}) \times_{M_n} Y(\Gamma)$, $\Gamma \in \Delta(n) - \{\gamma_n\}$. Let $D_\Gamma \subset X_\Gamma$, with $\Gamma \in \Delta(n)$, denote the exceptional Cartier divisor blown up from the strict transform of $Z(s_{canon}) \times_{M_n} Y(\Gamma)$ and denote the intermediate blown up scheme by X_Γ. Define $\tilde{X}$ to be the resulting scheme after blowing up all the (strict transforms of $Z(s_{canon}) \times_{M_n} Y(\Gamma)$, $\Gamma \in \Delta(n) - \{\gamma_n\}$) and the projection map $\tilde{X} \mapsto X$ can be factorized into the compositions of the various intermediate blowing down map.

[54]This ambiguity does not affect the result of our enumeration. see proposition 16 on page 75.

In the following discussion, we may pull back $\mathcal{O}(D_\Gamma)$ from X_Γ to $\tilde{X}$ from the various birational models (intermediate blowing ups) of X. To avoid complicated notations involving the line bundle or divisor pull-backs, we **skip the pull-back notations** consistently. The reader should be able to judge from the context of the formula and restore the pull-back notations accordingly.

At the end of this subsection, we introduce an index set $I_\Gamma \subset \Delta(n)$ collecting those Γ' smaller than $\Gamma \in \Delta(n)$ under $\models$.

Definition 9 *Let $\Gamma \in \Delta(n)$. The linear ordering $\models$ among all the $\Gamma \in \Delta(n)$ determines the ordering of the blowing ups to construct $\tilde{X}$ from X. Define I_Γ to be the subset of $\Delta(n)$ satisfying $I_\Gamma = \{\Gamma' | \Gamma \models \Gamma', \Gamma' \in \Delta(n)\}$.*

The index set $I_\Gamma, \Gamma \in \Delta(n) - \{\gamma_n\}$ collects all the admissible graphs Γ' in $\Delta(n) - \{\gamma_n\}$ whose associated zero loci $Z(s_{canon}) \times_{M_n} Y(\Gamma')$ (or more accurately their strict transformations) are blown up prior to $Z(s_{canon}) \times_{M_n} Y(\Gamma)$.

We notice that $I_{\gamma_n} = \Delta(n) - \{\gamma_n\}$. The collection of index sets $I_\Gamma$for $\Gamma \in \Delta(n)$ will be used to define the modified algebraic family invariant in the next sub-section.

5.2 The Definition of Modified Algebraic Family Invariants

In this subsection we define a version of modified algebraic family Seiberg-Witten invariant associated each $\Gamma \in \Delta(n)$. Recall that in section 5.3 on page 448 of [Liu1], a version of modified family Seiberg-Witten invariant has been defined in the differentiable category. The modified algebraic family Seiberg-Witten invariant we are going to define is its algebraic analogue.

The first step is to define a class $\tau_\Gamma \in K_0(Y(\Gamma) \times T(M))$, representable by a locally free sheaf (vector bundle) on the connected space $Y(\Gamma) \times T(M)$.

As usual we use $e_1, e_2, e_3, \cdots, e_n$ to denote the n type I exceptional classes over Y_Γ. Let $e_{k_i}, 1 \leq i \leq p$, be the type I exceptional classes over Y_Γ which pair negatively with the class $C - \mathbf{M}(E)E$.

As usual let $\Gamma_{e_{k_i}}$ denote the fan-like admissible graph such that the type I exceptional class e_{k_i} is effective and smooth/irreducible over the locally closed $Y_{\Gamma_{e_{k_i}}}$ (consult section 2 for more details).

Proposition 12 *Let $\tilde{\Xi}_{k_i} \mapsto Y(\Gamma_{e_{k_i}})$ be the relatively minimal $\mathbf{P}^1$ fiber bundle associated with the type I class $e_{k_i} = E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}}$.*

Suppose that $e_{k_i}^2 < e_{k_i} \cdot (C - \mathbf{M}(E)E) < 0$, then there exists an invertible sheaf $\mathcal{Q}_{k_i}$ over $\tilde{\Xi}_{k_i}$, pulled-back from $Y(\Gamma_{e_{k_i}})$, an effective relative divisor $\Delta_{k_i} \subset \tilde{\Xi}_{k_i} \mapsto Y(\Gamma_{e_{k_i}})$ of relative degree $-e_{k_i} \cdot (\mathbf{M}(E)E + e_{k_i})$ and the following short exact sequence of locally free sheaves,

$$0 \mapsto \mathcal{R}^0\tilde{\pi}_*\big(\mathcal{O}_{\Delta_{k_i}}(-m_{k_i}E_{k_i} - \sum_{j_{k_i}} m_{j_{k_i}}E_{j_{k_i}})\big) \otimes \mathcal{Q}_{k_i} \mapsto \mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}}(E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}})\big) \otimes \mathcal{Q}_{k_i}$$

$$\mapsto \mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}}(-m_{k_i}E_{k_i} - \sum_{j_{k_i}} m_{j_{k_i}}E_{j_{k_i}})\big) \otimes \mathcal{O}_{Y(\Gamma_{e_{k_i}})}(-\sum_{1\leq l<k_i} m_l E_{l;k_i}) \mapsto 0.$$

[55]

Suppose that $e_{k_i}^2 \geq e_{k_i} \cdot (C - \mathbf{M}(E)E)$, then there exists an invertible sheaf $\mathcal{Q}_{k_i}$ pulled-back from $Y(\Gamma_{e_{k_i}})$, an effective relative divisor $\Delta_{k_i} \subset \tilde{\Xi}_{k_i} \mapsto Y(\Gamma_{e_{k_i}})$ of relative degree $e_{k_i} \cdot (\mathbf{M}(E)E + e_{k_i})$ and the following short exact sequence of locally free sheaves on $Y(\Gamma_{e_{k_i}})$,

$$0 \mapsto \mathcal{R}^0\tilde{\pi}_*\big(\mathcal{O}_{\Delta_{k_i}}(E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}})\big) \otimes \mathcal{Q}_{k_i} \mapsto \mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}}(-m_{k_i}E_{k_i} - \sum_{j_{k_i}} m_{j_{k_i}}E_{j_{k_i}})\big) \otimes \mathcal{O}_{Y(\Gamma_{e_{k_i}})}(-\sum_{1\leq l<k_i} m_l E_{l;k_i})$$

$$\mapsto \mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}}(E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}})\big) \otimes \mathcal{Q}_{k_i} \mapsto 0.$$

Proof: The proof is almost identical to the proof of proposition 15 in [Liu5] and the reader can consult the cited paper for its derivation. Instead of using the $\mathbf{P}^1$ fibrations $\Xi_{k_i} \mapsto Y(\Gamma_{e_{k_i}})$, we use the $\mathbf{P}^1$ fiber bundles $\tilde{\Xi}_{k_i}$. Because the details of the argument is almost identical, we omit it here. □

The following lemma characterizes the significance of the locally free sheaf $\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}}(E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}})\big)$.

Lemma 16 *Let $\mathcal{N}_{Y(\Gamma_{e_{k_i}})}$ denote the normal sheaf of $Y(\Gamma_{e_{k_i}}) \subset M_n$. Then there exists a canonical isomorphism $\mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}}(E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}})\big) \mapsto \mathcal{N}_{Y(\Gamma_{e_{k_i}})}$.*

Outline of the Proof: The key idea is to study the canonical algebraic family Kuranishi model of e_{k_i},

$$0 \mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{M_{n+1}}(E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}})\big) \mapsto \mathcal{R}^0\pi_*\mathcal{O}_{M_{n+1}}(E_{k_i})$$

$$\mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum_{j_{k_i}} E_{j_{k_i}}}(E_{k_i})\big) \mapsto \mathcal{R}^1\pi_*\big(\mathcal{O}_{M_{n+1}}(E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}})\big) \mapsto 0.$$

Then the lemma is a direct consequence of lemma 9 in section 6.1 of [Liu5], once we realize that the birational projection (see lemma 7) $\Xi_{k_i} \mapsto \tilde{\Xi}_{k_i}$ induces an isomorphism $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\Xi_{k_i}}(E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}})\big) \stackrel{\cong}{\mapsto} \mathcal{R}^1\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}}(E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}})\big)$. □

[55]The symbol $E_{a;b}$, $a < b$, denotes the exceptional divisor in M_n by blowing up the strict transform of the $(a, b)-$th partial diagonal.

If at least one of e_{k_i} satisfies $e_{k_i}^2 < e_{k_i} \cdot (C - \mathbf{M}(E)E) < 0$, then we define $\tau_\Gamma \equiv 0 \in K_0(Y(\Gamma) \times T(M))$ following the rationale of theorem 3. of [Liu5] and case II of the proof of proposition 18 starting at page 79. From now on we may assume that $0 > e_{k_i}^2 \geq e_{k_i} \cdot (C - \mathbf{M}(E)E)$ for all $1 \leq i \leq p$.

Recall that it was defined in subsection 3.1 definition 3 that the index subset I_{k_l} collects the indexes in $\{1, 2, \cdots, n\}$ occurring in $e_{k_l} = E_{k_l} - \sum_{j_{k_l}} E_{j_{k_l}}$, i.e. k_l and all its direct descendent indexes in the graph Γ.

We define $\tau_\Gamma \in K_0(Y(\Gamma) \times T(M))$ as the following,

Definition 10 *Let $e_{k_i}, 1 \leq i \leq p$ denote the type I exceptional classes among $e_1, e_2, \cdots, e_n$ over $Y(\Gamma)$ which have negative pairings with $C - \mathbf{M}(E)E$. Suppose that $e_{k_i}^2 \geq e_{k_i} \cdot (C - \mathbf{M}(E)E)$ for all $1 \leq i \leq p$, define $\tau_\Gamma \equiv [\oplus_{1 \leq l \leq p} \mathcal{R}^1 \tilde{\pi}_* \big(\mathcal{O}_{\tilde{\Xi}_{k_l}} \otimes \mathcal{E}_{C - \sum_{a \in I_{k_l}} m_a E_a - \sum_{p \geq a > l} e_{k_a}}\big) \otimes \mathcal{O}_{Y(\Gamma)}(-\sum_{1 \leq r < k_l} m_r E_{r;k_l}) - \oplus_{1 \leq l \leq p} \mathcal{R}^1 \tilde{\pi}_* \big(\mathcal{O}_{\tilde{\Xi}_{k_l}}(E_{k_l} - \sum_{j_{k_l}} E_{j_{k_l}})\big) \otimes \mathcal{Q}_{k_l} \otimes \mathcal{E}_C] \in K_0(Y(\Gamma) \times T(M))$. Otherwise[56], set τ_Γ to be zero.*

Please compare the definition of τ_Γ with definition 4 of $\tilde{\mathcal{V}}_{quot}$ on page 36.

Lemma 17 *The element τ_Γ can be represented by a locally free sheaf of rank $\sum_{1 \leq l \leq p} e_{k_l} \cdot (e_{k_l} + \mathbf{M}(E)E + \sum_{1 \leq j < l} e_{k_j})$,*

$$\oplus_{1 \leq l \leq p} (\mathcal{R}^0 \tilde{\pi}_* \big(\mathcal{O}_{\Delta_{k_l}}(E_{k_l} - \sum_{j_{k_l}} E_{j_{k_l}})\big) \otimes \mathcal{Q}_{k_l} \otimes \mathcal{E}_C \oplus_{1 \leq l < t \leq p} \mathcal{R}^0 \tilde{\pi}_* \big(\mathcal{O}_{\tilde{\Xi}_{k_l} \cap \tilde{\Xi}_{k_t}}(-\sum_{a \in I_{k_l}} m_a E_a) \otimes \mathcal{E}_C\big)) \otimes \mathcal{O}_{Y(\Gamma)}(-\sum_{1 \leq r < k_l} m_r E_{r;k_l}).$$

The symbol $\tilde{\Xi}_{k_l} \cap \tilde{\Xi}_{k_t}$ used here denotes the cross section of $\tilde{\Xi}_{k_l}|_{Y(\Gamma)} \mapsto Y(\Gamma)$ (or $\tilde{\Xi}_{k_t}|_{Y(\Gamma)} \mapsto Y(\Gamma)$ induced by E_{k_t} (or E_{k_l}) whenever $e_{k_l} \cdot e_{k_t} = 1$. It is taken to be the empty set when $e_{k_l} \cdot e_{k_t} = 0$.
Proof: The lemma is a direct consequence of a collection of $\mathcal{O}_{Y(\Gamma)}(-\sum_{1 \leq r < k_l} m_r E_{r;k_l})$ twisted version of short exact sequences for different l,

$$0 \mapsto \oplus_{p \geq i > l \geq 1} \mathcal{R}^0 \tilde{\pi}_* \big(\mathcal{O}_{\tilde{\Xi}_{k_l} \cap \tilde{\Xi}_{k_i}} \otimes \mathcal{E}_{C - \sum_{a \in I_{k_l}} m_a E_a}\big) \mapsto \mathcal{R}^1 \tilde{\pi}_* \big(\mathcal{O}_{\tilde{\Xi}_{k_l}} \otimes \mathcal{E}_{C - \sum_{a \in I_{k_l}} m_a E_a - \sum_{p \geq t > l} e_{k_t}}\big)$$

$$\mapsto \mathcal{R}^1 \tilde{\pi}_* \big(\mathcal{O}_{\tilde{\Xi}_{k_l}} \otimes \mathcal{E}_{C - \sum_{a \in I_{k_l}} m_a E_a}\big) \mapsto 0$$

for all l ranging in $1 \leq l \leq p$ and the $\mathcal{E}_C$-twisted versions of the short exact sequences in proposition 12. The above sheaf short exact sequences are the derived exact sequences of sheaf short exact sequences on $\tilde{\Xi}_{k_l}$, $1 \leq l \leq p$, of the divisors $\cup_{p \geq t > l} \tilde{\Xi}_{k_l} \cap \tilde{\Xi}_{k_t} \subset \tilde{\Xi}_{k_l}$. They truncate to short exact sequences because $\mathcal{R}^0 \tilde{\pi}_* \big(\mathcal{O}_{\tilde{\Xi}_{k_l}} \otimes \mathcal{E}_{C - \sum_{a \in I_{k_l}} m_a E_a}\big) = 0$, due to the negativity of the relative degrees $e_{k_l} \cdot (C - \mathbf{M}(E)E) = e_{k_l} \cdot (C - \sum_{a \in I_{k_l}} m_a E_a)$ on the $\mathbf{P}^1$ fiber bundles.

[56] If some e_{k_l} satisfies $e_{k_l}^2 \leq e_{k_l} \cdot (C - \mathbf{M}(E)E) < 0$.

The calculation on its rank follows from $deg\big(\Delta_{k_i}/Y(\Gamma_{e_{k_i}})\big) = e_{k_i}\cdot(\mathbf{M}(E)E+e_{k_i}) = e_{k_i}\cdot(\sum_{a\in I_{k_i}} m_aE_a+e_{k_i})$ and $deg(\big(\sum_{p\geq i>l\geq 1}\tilde{\Xi}_{k_i}\big)\cap\tilde{\Xi}_{k_l}/Y(\Gamma)) = \sum_{p\geq i>l\geq 1} e_{k_i}\cdot e_{k_l}$.

The explicit representative of τ_Γ is locally free because each summand is a zero-th derived image sheaf along a finite morphism onto $Y(\Gamma)$. $\Box$

The inductive definitions of the modified algebraic family Seiberg-Witten invariants are parallel to the induction procedure in enumerating the local contributions of the family invariants. The reader who wants to find out the geometric motivation for our definition may consult section 6 for the parallelism. On the other hand, the current inductive scheme is also parallel to the definition of modified family Seiberg-Witten invariants in the differentiable category. The reader may consult subsection 5.3 of [Liu1] for more details.

Recall (see section 6.5 of [Liu5]) the following definition of the partial ordering $\gg$ among the pairs $(\Gamma,\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i)$.

Definition 11 *Let $\Gamma>\Gamma'$ be two n-vertex admissible graphs and let e_i, e_i', $1\leq i\leq n$ denote the type I exceptional classes associated with Y_Γ, $Y_{\Gamma'}$, respectively. The pair $(\Gamma,\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i)$ is said to be greater than $(\Gamma',\sum_{e_i'\cdot(C-\mathbf{M}(E)E)<0}e_i')$ under $\gg$, denoted as*

$$(\Gamma,\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i)\gg(\Gamma',\sum_{e_i'\cdot(C-\mathbf{M}(E)E)<0}e_i'),$$

if the following conditions hold.
(A). For all the indexes i, $1\leq i\leq n$ such that the type I classes over Y_Γ, e_i, satisfy $e_i\cdot(C-\mathbf{M}(E)E)<0$, then $e_i'=e_i$.
and
(B). There $\exists$ at least one e_j' over $Y_{\Gamma'}$, $e_j'\cdot(C-\mathbf{M}(E)E)<0$ but the corresponding e_j over Y_Γ with the same subscript j satisfies $e_j\cdot(C-\mathbf{M}(E)E)\geq 0$.

The conditions (A). and (B). imply that some new type I class which pairs negatively with $C-\mathbf{M}(E)E$ shows up above the sub-locus $Y_{\Gamma'}\subset Y_\Gamma$ while the original negative $C-\mathbf{M}(E)E-$paired e_i persists to be irreducible over $Y_{\Gamma'}$ and do **not break up**.

Refer to fig.5 below for an example[57].

We abbreviate the above partial order relationship by $\Gamma\gg\Gamma'$ if a multiplicity function $\mathbf{M}(E)E$ has been fixed throughout the discussion.

We have the following simple observation regarding $\gg$ and $\succ$.

Lemma 18 *Let $\Gamma\gg\Gamma'$, then $\Gamma\succ\Gamma'$.*

Proof: The cone $\mathcal{C}_\Gamma$ are generated by the effective type I classes e_i such that $e_i\cdot(C-\mathbf{M}(E)E)<0$ and some other type I -1 classes. By our assumption on $\Gamma\gg\Gamma'$, these e_i persist to become e_i' over $Y_{\Gamma'}$ with $e_i'\cdot(C-\mathbf{M}(E)E)<0$.

[57]To simplify the notations, the $\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i$ part has been skipped. this causes no problem when a $\mathbf{M}(E)E$ is fixed throughout the discussion.

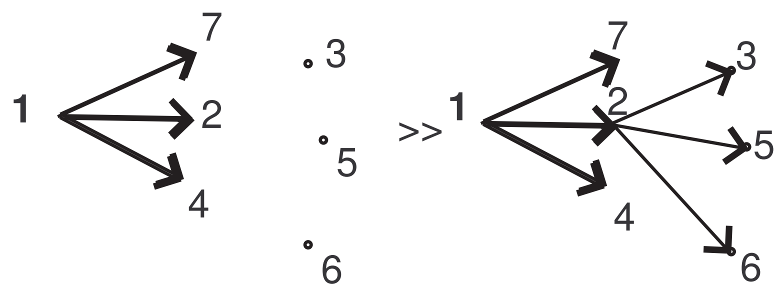

fig.5
A pair of admissible graphs related by $\gg$, the graph on the right hand side is a degeneration from the left hand side such that E_2 breaks into $E_2 - E_3 - E_5 - E_6$ and the union of E_3, E_5, E_6.

As these e'_i are a subset of the generators of $\mathcal{C}_{\Gamma'}$, this implies that $\mathcal{C}_{\Gamma'} \supset \mathcal{C}_{\Gamma}$. Therefore $\Gamma \succ \Gamma'$. $\Box$

The set $\Delta(n)$ is a finite set. Therefore there must be minimal elements (may be non-unique) under the partial ordering $\gg$.

Definition 12 *Let $\Gamma \in \Delta(n)$ be a minimal element under $\gg$. Define the modified algebraic family invariant $\mathcal{AFSW}^*_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$ to be $\mathcal{AFSW}_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E - \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$, where $c_{total}(\tau_\Gamma)$ is the total Chern class of $\tau_\Gamma \in Y(\Gamma)\times T(M)$ defined in definition 10.*

Let Γ be in $\Delta(n)$. Suppose that for all the elements in $\Delta(n)$ smaller than $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$ under $\gg$, the modified algebraic family invariants have been defined already. Set the modified invariant attached to Y_Γ, $\mathcal{AFSW}^*_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$ to be,

Definition 13 *Let $\tau_\Gamma, \tau_{\Gamma'}$ be the K_0 theory classes by definition 10 associated with Γ, Γ', respectively. Define $\mathcal{AFSW}^*_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E - \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$ to be*

$$\mathcal{AFSW}_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$$

$$-\sum_{\Gamma\gg\Gamma'} \mathcal{AFSW}^*_{M_{n+1}\times_{M_n}Y(\Gamma')\times T(M)\mapsto Y(\Gamma')\times T(M)}(c_{total}(\tau_{\Gamma'}), C-\mathbf{M}(E)E$$

$$-\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i).$$

One may argue easily that the procedure of the inductive definition always continues until all the elements in $\Delta(n)$ are exhausted. Suppose that the process halts before exhausting the elements in $\Delta(n)$. Namely, there exists no $\Gamma \in \Delta(n)$ such that the modified algebraic family invariants have been defined for all $(\Gamma', \sum_{e_i'\cdot(C-\mathbf{M}(E)E)<0} e_i') \ll (\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$. If it is the case, then for any $\Gamma \in \Delta(n)$ that the modified algebraic family Seiberg-Witten invariant is not defined yet, one must be able to find at least one $\Gamma' \in \Delta(n)$ such that $(\Gamma', \sum_{e_i'\cdot(C-\mathbf{M}(E)E)<0} e_i') \ll (\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$ and the modified algebraic family invariant is not defined for $(\Gamma', \sum_{e_i'\cdot(C-\mathbf{M}(E)E)<0} e_i')$, either. Then one may trace along the smaller and smaller elements under $\gg$ (the procedure involves choices and may not be canonical) and it has to stop after a finite number of steps since $\Delta(n)$ is a finite set. But such a terminal graph Γ has to be a minimal element under $\gg$ and its modified algebraic family invariant has been defined in definition 12 already. This generates a contradiction and thus the above procedure never halts unless all the elements in $\Delta(n)$ has been exhausted.

After a finite number of steps and the defining process has to terminate at $\gamma_n \in \Delta(n)$. In this case one defines $\mathcal{AFSW}^*_{M_{n+1}\times T(M)\mapsto M_n\times T(M)}(1, C-\mathbf{M}(E)E)$ by the following recipe,

Definition 14 *Define* $\mathcal{AFSW}^*_{M_{n+1}\times T(M)\mapsto M_n\times T(M)}(1, C-\mathbf{M}(E)E)$ *to be*

$$\mathcal{AFSW}_{M_{n+1}\times T(M)\mapsto M_n\times T(M)}(1, C-\mathbf{M}(E)E) - \sum_{\Gamma\in\Delta(n)-\{\gamma_n\}} \mathcal{AFSW}^*(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E - \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i).$$

Definition 14 can be viewed as an extension of definition 13 once we realize that for $\gamma \in adm(n)$, $Y(\gamma_n) = M_n$, $\gamma_n > \Gamma$ for all $\Gamma \in \Delta(n) - \{\gamma_n\}$ and $(\gamma_n, 0) \gg (\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$ for all $\Gamma \in \Delta(n) - \{\gamma_n\}$.

Remark 12 *Suppose that $t_L \in T(M)$ is a point of the connected component $T(M)$ (determined by the first Chern class C) of the Picard variety $Pic(M)$. There is a corresponding version of "t_L restricted" modified algebraic family Seiberg-Witten invariants defined by inserting $[t_L] \in \mathcal{A}_0(T(M))$ into each terms in the definition. The resulting modified invariant is denoted by*

$$\mathcal{AFSW}^*_{M_{n+1}\times_{M_n} Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma)\cap[t_L], C-\mathbf{M}(E)E - \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i),$$

or equivalently

$$\mathcal{AFSW}^*_{M_{n+1}\times_{M_n} Y(\Gamma)\times \{t_L\}\mapsto Y(\Gamma)\times\{t_L\}}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E - \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i).$$

When the point $t_L \in T(M)$ is determined by an algebraic line bundle $L \mapsto M$ with $c_1(L) = C$, the "t_L-restricted" modified algebraic family invariants enumerate the curves dual to $C - \mathbf{M}(E)E$ resolved from the linear subsystem of $|L|$.

Remark 13 *In the earlier paper [Liu5], we had shown that under the* **Special Condition***, the dominated localized top Chern class contribution of $Y(\Gamma)$ is nothing but the mixed algebraic family Seiberg-Witten invariant*

$$\mathcal{AFSW}_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i).$$

When the **Special Condition** *is not met, our definitions of the modified invariants indicates that there are correction terms captured by the partial ordering $\gg$ besides the dominated term.*

A key proposition in proving the main theorem of the paper is the following,

Proposition 13 *Let $\Gamma \in \Delta(n)$, then the modified algebraic family Seiberg-Witten invariant $\mathcal{AFSW}^*_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$ can be expressed as a homogeneous universal polynomial of $C^2 = C\cdot C$, $C\cdot c_1(\mathbf{K}_M)$, $c_1(\mathbf{K}_M)^2$ and $c_2(M)$ of degree n multiplied by $\mathcal{ASW}(C)$. The universal polynomial depends on the graph Γ and the singular multiplicities $\mathbf{M}(E)E$ but does not depend upon the algebraic surface M.*

When $\Gamma = \gamma_n$, we set $c_{total}(\tau_{\gamma_n}) = 1$.
Proof: As all of the modified invariants $\mathcal{AFSW}^*_{M_{n+1}\times T(M)\mapsto M_n\times T(M)}(1, C-\mathbf{M}(E)E)$ and $\mathcal{AFSW}^*_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$, $\Gamma \in \Delta(n) - \{\gamma_n\}$, are defined by inductive procedures based on the mixed algebraic invariants, we prove that $\mathcal{AFSW}_{M_{n+1}\times T(M)\mapsto M_n\times T(M)}(1, C-\mathbf{M}(E)E)$ and all the $\mathcal{AFSW}_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$, $\Gamma \in \Delta(n)$, can be expressed as universal (independent of M) homogeneous polynomials of degree n of $C\cdot C$, $C\cdot c_1(\mathbf{K}_M)$, $c_1(\mathbf{K}_M)\cdot c_1(\mathbf{K}_M)$, and $c_2(M)$, multiplied by the algebraic Seiberg-Witten invariant $\mathcal{ASW}(C)$ of C.

We present the detailed argument for $\mathcal{AFSW}_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$, $\Gamma \in \Delta(n) - \{\gamma_n\}$ and the proof for the case of $\mathcal{AFSW}_{M_{n+1}\times T(M)\mapsto M_n\times T(M)}(1, C-\mathbf{M}(E)E)$ is essentially parallel.

Recall[58] that the fiber bundle projection map $f_n : M_{n+1} \mapsto M_n$ can be constructed from $M_n \times M \mapsto M_n$ through n consecutive blowing ups along (codimension two) cross sections of the intermediate fiber bundles. This implies that its pull-back to $Y(\Gamma)$, $M_{n+1}\times_{M_n}Y(\Gamma) \mapsto Y(\Gamma)$, can be constructed from the Cartesian projection $Y(\Gamma)\times M \mapsto Y(\Gamma)$ through n consecutive blowing ups along cross sections of the intermediate blown up spaces. Schematically this implies

[58] Consult lemma 3.1 on page 401 of [Liu1].

that we may apply the family blowup formula of the algebraic family Seiberg-Witten invariants [Liu3] to relate $\mathcal{AFSW}_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i)$ and the mixed algebraic family Seiberg-Witten invariant $\mathcal{AFSW}_{Y(\Gamma)\times T(M)\times M\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma)\cap c_{tatal}(\mathbf{U}_{\mathbf{M}(E)}), C)$. The bundle $\mathbf{U}_{\mathbf{M}(E)}$ appearing in the identity is the relative obstruction bundle

$$\mathbf{U}_{\mathbf{M}(E)E}=\oplus_{1\leq l\leq n}\mathcal{E}_{C-\sum_{1\leq a\leq l-1}m_aE_a}\otimes\mathbf{S}^{m_l-1}(\mathbf{C}\oplus(f_{n-1;l}|^*_{Y(\Gamma)}\mathbf{T}^*_{M_l/f^*_{l-1}M_{l-1}}))$$

, gotten from applying the algebraic family blowup formula n times from $M_n\times M\mapsto M_n$ to $M_{n+1}\mapsto M_n$. The map $f_{n-1;l}|_{Y(\Gamma)}:Y(\Gamma)\mapsto M_l$ is the composition $Y(\Gamma)\subset M_n\stackrel{f_{n-1;l}}{\longrightarrow}M_l$.

We have the following simple factorization lemma regarding the family invariants,

Lemma 19 *Let $M\times B\mapsto B$ be a product algebraic fiber bundle over a complete and smooth base B and let $\underline{C}$ be a $(1,1)$ class on the algebraic surface M, then $\mathcal{AFSW}_{B\times M\mapsto B}(\eta,\underline{C})=0$ for $\eta\notin\mathcal{A}_0(B)$ and is equal to $\mathcal{ASW}(\underline{C})\cdot(\int_B\eta)$ for $\eta\in\mathcal{A}_0(B)$.*

Proof of lemma 19: For simplicity we assume that $\mathcal{E}_{\underline{C}}$ has a vanishing second derived image sheaf[59] over $T(M)\times B$. Consider the algebraic family moduli space of $\underline{C}$, $\mathcal{M}_{\underline{C}}$, over B. Because the fiber bundle of algebraic surfaces is a trivial product, the space $\mathcal{M}_{\underline{C}}$ is also a trivial product over B and the algebraic family Kuranishi models of $\underline{C}$ are pulled back from $T(M)$ to $T(M)\times B$. Let $(\mathbf{V},\mathbf{W},\Phi_{\mathbf{VW}})$ be one algebraic family Kuranishi model of $\underline{C}$, where $\mathbf{V},\mathbf{W}$ are vector bundles over $T(M)\times B$ pulled-back from $T(M)$.

Then for $\eta\in\mathcal{A}_k(B)$, $k\leq dim_{\mathbf{C}}B$, the mixed family invariant $\mathcal{AFSW}_{B\times M\mapsto B}(\eta,\underline{C})$ can be expressed as the push-forward of

$$\int_{\mathbf{P}(\mathbf{V})}c_1(\mathbf{H})^{rank_{\mathbf{C}}(\mathbf{V}-\mathbf{W})-1+q(M)+k}\cap c_{top}(\mathbf{W}\otimes\mathbf{H})\cap\eta\cap[\mathbf{P}(\mathbf{V})]\in\mathcal{A}_0(pt)\cong\mathbf{Z},$$

into $\mathcal{A}_0(pt)\cong\mathbf{Z}$.

Because $\mathbf{P}(\mathbf{V})$ is also a trivial product over B, $c_1(\mathbf{H})^{rank_{\mathbf{C}}(\mathbf{V}-\mathbf{W})+q(M)-1+k}\cap c_{top}(\mathbf{W}\otimes\mathbf{H})\cap[\mathbf{P}(\mathbf{V})]=0$ for all $k>0$. On the other hand, when $\eta\in\mathcal{A}_0(B)$ the mixed invariant can be expressed as (for some $b\in B$)

$$\int_{\mathbf{P}(\mathbf{V}|_{T(M)\times\{b\}})}c_1(\mathbf{H})^{rank_{\mathbf{C}}(\mathbf{V}-\mathbf{W})-1+q(M)}\cap c_{top}(\mathbf{W}|_{T(M)\times\{b\}}\otimes\mathbf{H})\cap\mathbf{P}(\mathbf{V}|_{T(M)\times\{b\}})\cdot\int_B\eta=\mathcal{ASW}(\underline{C})\cdot\int_B\eta.$$

The case when the second derived image sheaf of $\mathcal{E}_{\underline{C}}$ is not vanishing can be discussed similarly and we omit the details here. $\Box$

[59]When we apply this lemma to the concrete situation below, this additional assumption is satisfied.

By applying lemma 19 to our context, the above mixed invariant is equal to $\mathcal{ASW}(C)\cdot\int_{M_n}\{c_{total}(\tau_\Gamma)\cap c_{total}(\mathbf{U}_{\mathbf{M}(E)})\cap[Y(\Gamma)]\}_0$.

Our goal is to show that the intersection number $\int_{M_n}\{c_{total}(\tau_\Gamma)\cap c_{total}(\mathbf{U}_{\mathbf{M}(E)})\cap[Y(\Gamma)]\}_0$ is a universal homogeneous polynomial of degree n in terms of the Chern numbers $C^2[M]$, $C\cdot c_1(\mathbf{K}_M)[M]$, $c_1(\mathbf{K}_M)^2[M]$ and $c_2(M)[M]$. If for at least one type I class e_{k_i}, $1\leq i\leq p$, the inequality $e_{k_i}^2 < e_{k_i}\cdot(C-\mathbf{M}(E)E)$ holds, then $\tau_\Gamma\equiv 0$ and the mixed algebraic family invariant over $Y(\Gamma)$ has been defined to be zero.

So we may assume that $e_{k_i}^2\geq e_{k_i}\cdot(C-\mathbf{M}(E)E)$ for all $1\leq i\leq p$. Then by lemma 17 the explicit expression of the class τ_Γ enables to conclude that,

Lemma 20 *Let $\pi_i: M_n\mapsto M$ be the composite projection map $M_n\mapsto M^n\mapsto M$ to the i-th copy of M, $1\leq i\leq n$ and let*[60] *$E_{a;b}$ (for $a<b$) denote the exceptional divisor of $M_n\mapsto M^n$ associated with the (a,b)-th partial diagonal of M^n.*

The image of the total Chern class $c_{total}(\tau_\Gamma)$ under $\mathcal{A}.(Y(\Gamma))\mapsto\mathcal{A}.(M_n)$ can be expressed as a universal degree $rank_{\mathbf{C}}\tau_\Gamma$ polynomial of the cycle classes $\pi_i^*C\in\mathcal{A}_{2n-2}(M_n)$, $E_{a;b}\in\mathcal{A}_{2n-2}(M_n)$, for $a<b$.

Proof: By the locally free representative of τ_Γ in lemma 17, one may write

$$c_{total}(\tau_\Gamma)=c_{total}(\oplus_{1\leq l\leq p}(\mathcal{R}^0\tilde{\pi}_*\big(\mathcal{O}_{\Delta_{k_l}}(E_{k_l}-\sum_{j_{k_l}}E_{j_{k_l}})\big)\otimes\mathcal{Q}_{k_l}\otimes\mathcal{E}_C)\oplus_{p\geq t>l\geq 1}\mathcal{R}^0\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_l}\cap\tilde{\Xi}_{k_t}}(-\sum_{a\in I_{k_l}}m_aE_a)\otimes\mathcal{E}_C\big))).$$

It suffices to show that the total Chern class of each of the locally free sheaves $\mathcal{R}^0\tilde{\pi}_*\big(\mathcal{O}_{\Delta_{k_l}}(E_{k_l}-\sum_{j_{k_l}}E_{j_{k_l}})\big)\otimes\mathcal{Q}_{k_l}\otimes\mathcal{E}_C$ and $\mathcal{R}^0\tilde{\pi}_*\big(\mathcal{O}_{\tilde{\Xi}_{k_l}\cap\tilde{\Xi}_{k_t}}(-\sum_{a\in I_{k_l}}m_aE_a)\otimes\mathcal{E}_C\big)$ is a polynomial in terms of all the flat pull-back π_i^*C and all the $E_{i;j}$, $i<j$.

Firstly recall how the invertible sheaf $\mathcal{E}_C\mapsto T(M)\times M_{n+1}$, $n\in\mathbf{N}$, has been constructed.

Choose a point $t_{L_0}\in T(M)$ which corresponds to a invertible sheaf $\mathcal{L}_0\mapsto M$ with $c_1(\mathcal{L}_0)=C$. Consider the universal invertible sheaf $\mathcal{L}_{univ}\mapsto T(M)\times M$, then $\mathcal{L}_{univ}\otimes\pi^*_{T(M)\times M\mapsto M}\mathcal{L}_0$ defines the invertible sheaf $\mathcal{E}_C$ over $T(M)\times M$. To pull it back to $T(M)\times M_{n+1}$, we consider the projection $T(M)\times M_{n+1}\mapsto T(M)\times M^{n+1}$ to the trivial product. By composing it with $T(M)\times M^{n+1}\mapsto T(M)^{n+1}\times M^{n+1}\cong(T(M)\times M)^{n+1}$ induced by $T(M)\ni\{t\}\mapsto\{t\}\times\{t_{L_0}\}\times\cdots\{t_{L_0}\}\in T(M)^{n+1}$, the pulled-back invertible sheaf is what we denote as $\mathcal{E}_C$ throughout this paper. It is easy to see that the construction is independent to the choices of $t_{L_0}\in T(M)$.

Fix the $\mathbf{P}^1$ fiber bundle $\tilde{\Xi}_{k_l}\mapsto Y(\Gamma_{e_{k_l}})$ for some $1\leq l\leq p$, the relative divisors $\Delta_{k_l}\mapsto Y(\Gamma_{e_{k_l}})$ and $\tilde{\Xi}_{k_l}\cap\tilde{\Xi}_{k_t}=E_{k_t}|_{\tilde{\Xi}_{k_l}}$ are (multiples of) cross sections of the given $\mathbf{P}^1$ fiber bundle $\tilde{\Xi}_{k_l}\mapsto Y(\Gamma_{e_{k_l}})$. For every direct descendent index j_{k_l} of k_l in the admissible graph Γ, the exceptional divisor $E_{j_{k_l}}$ determines

[60] They were denoted as $E_a(b)\in H^2(M_n,\mathbf{Z})$ at page 402, proposition 3.1 of [Liu1], in the topological category.

a cross-section of $\tilde{\Xi}_{k_l} \mapsto Y(\Gamma_{e_{k_l}})$ and the restriction of the invertible sheaf $\mathcal{O}_{\tilde{\Xi}_{k_l}}(E_j)$, $j \neq j_{k_l}$, to this cross section determined by $E_{j_{k_l}}$ is isomorphic to the pull-back of $\mathcal{O}_{Y(\Gamma_{e_{k_l}})}(E_{min(j_{k_l}.j);max(j_{k_l}.j)})$ from the base.

It is easy to see by a simple induction argument that the derived image sheaf $\mathcal{R}^0\tilde{\pi}_*(\mathcal{O}_{mE_{j_{k_l}}}|_{\tilde{\Xi}_{k_l}}) \cong \oplus_{0\leq i\leq m-1}(\mathcal{O}_{Y(\Gamma_{e_{k_l}})}(E_{k_l;j_{k_l}}))^{\otimes -i}$ for any direct descendent index j_{k_l} of k_l. On the other hand, $c_1(\mathcal{E}_C|_{E_{j_{k_l}}\cap\tilde{\Xi}_{k_l}}) = c_1(\pi_{k_l}^*\mathcal{L}_0|_{Y(\Gamma_{e_{k_l}})}) = \pi_{k_l}^*C|_{Y(\Gamma_{e_{k_l}})}$. By combining these ingredients, the total Chern class of τ_Γ can be determined and must be an M-independent polynomial in terms of the various $\pi_i^*C, E_{i;j}$. etc. $\square$

Remark 14 *If one pulls back the invertible sheaf from $\mathcal{L}_{univ}\otimes\mathcal{L}_0 \mapsto T(M)\times M$ by $T(M)\times M_{n+1} \mapsto (T(M)\times M)^{n+1}$ which instead factors through the diagonal embedding $T(M)\times M^{n+1} \overset{\Delta_{T(M)\times id_{M^{n+1}}}}{\longrightarrow} T(M)^{n+1}\times M^{n+1}$, then this invertible sheaf differs from our $\mathcal{E}_C$ by an invertible sheaf pulled-back from the base $T(M)\times M_n$. If one adopts this alternative invertible sheaf and calculates its first Chern class, it will involves not only $c_1(\mathcal{L}_0)$ but also $c_1(\mathcal{L}_{univ})$. On the other hand, the algebraic family Kuranishi model it determines can be gotten from $(\mathcal{V}_{canon}, \mathcal{W}_{canon}, \Phi_{\mathcal{V}_{canon}\mathcal{W}_{canon}})$ by twisting the invertible sheaf pulled-back from $T(M)\times M_n$. It is easy to see that the final answer of $\mathcal{AFSW}^*_{T(M)\times M_{n+1}\mapsto T(M)\times M_n}(1, C-\mathbf{M}(E)E)$ is independent of the twisting on the algebraic family Kuranishi models. Our choice of $\mathcal{E}_C$ has the benefit of separating the contribution of $c_1(\mathcal{L}_{univ})$ to the family invariant through the factor $\mathcal{ASW}(C)$ (see remark 15).*

Let $\Gamma \in \Delta(n)$. Then the type I exceptional classes e_{k_i}, $1 \leq i \leq p$, are effective over $Y(\Gamma)$ and the type I exceptional classes e_j, $j \notin \{k_1, k_2, k_3, \cdots, k_p\}$ are -1 classes. Recall from proposition 4 of section 2 that the smooth locus $Y(\Gamma)$ is the transversal intersection of $Y(\Gamma_{e_{k_i}})$, $1 \leq i \leq p$, where $\Gamma_{e_{k_i}}$ is the fan-like admissible graph associated with e_{k_i} (see section 2). Then $Y(\Gamma) = \cap_{1\leq i\leq p} Y(\Gamma_{e_{k_i}})$ and in $\mathcal{A}_{dim_{\mathbf{C}}Y(\Gamma)}(M_n)$ we have the equality of cycle classes $[Y(\Gamma)] = \cap_{i\leq p}[Y(\Gamma_{e_{k_i}})]$.

Recall $codim_{\mathbf{C}}\Gamma$ is the number of one-edges in Γ. Each $[Y(\Gamma_{e_{k_i}})]$ is an algebraic cycle class of dimension $dim_{\mathbf{C}}M_n + \frac{e_{k_i}\cdot e_{k_i} - c_1(\mathbf{K}_{M_{n+1}/M_n})\cdot e_{k_i}}{2} = 2n - codim_{\mathbf{C}}\Gamma_{e_{k_i}}$. To calculate the cycle class explicitly, there are essentially two equivalent methods. Either we consider the canonical algebraic family Kuranishi model of $e_{k_i} = E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}}$ as was done in section 6.2, lemma 9 of [Liu5], and $Y(\Gamma_{e_{k_i}})$ is the regular zero locus of the canonical obstruction bundle, so $[Y(\Gamma_{e_{k_i}})] \in \mathcal{A}_{2n-codim_{\mathbf{C}}\Gamma_{e_{k_i}}}(M_n)$ represents the top Chern class of the obstruction bundle and can be determined explicitly. Or one may apply the algebraic family blow up formula to the class $e_{k_i} = E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}}$, one may find the top Chern class of the obstruction bundle inductively. By either means the answer of $[Y(\Gamma_{e_{k_i}})]$ is expressible as

$$\bigcap_{1\leq s\leq codim_{\mathbf{C}}\Gamma_{e_{k_i}}}(E_{k_i;j^s_{k_i}}-\sum_{r\leq s-1}E_{j^r_{k_i};j^s_{k_i}})\in \mathcal{A}_{2n-codim_{\mathbf{C}}\Gamma_{e_{k_i}}}(M_n),$$

where $j^1_{k_i}<j^2_{k_i}<j^3_{k_i}<\cdots<j^{codim_{\mathbf{C}}\Gamma_{e_{k_i}}}_{k_i}$ are the direct descendent indexes of k_i.

By combining these calculations together, we find that the mixed family invariant $\mathcal{AFSW}_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_{k_i}\cdot(C-\mathbf{M}(E)E)<0}e_{k_i})$ can be expressed as $\mathcal{ASW}(C)$ times the $\int_{M_n}$ of a polynomial expression of the various $E_{i;j}$, $1\leq i<j\leq n$, $c_{total}(\mathbf{T}M_n)$, $c_{total}(f^*_{n-1,k}\mathbf{T}M_k)$, and the various π_i^*C.

To determine the push-forward of the zero cycle class under $\mathcal{A}_0(M_n)\mapsto\mathcal{A}_0(pt)$, observe that the morphism factors through $\mathcal{A}_0(M_n)\mapsto\mathcal{A}_0(M^n)\mapsto\mathcal{A}_0(pt)\cong\mathbf{Z}$. We notice the following facts:

(i). the projection $M_n\mapsto M^n$ can be factorized as $\frac{n(n-1)}{2}$ consecutive codimension-two blowing down maps,

and

(ii). the well known blowup formula of Chern classes(see page 298, section 15.4 of [F]),

and the fact that

(iii). the exceptional divisor of a codimension-two blowing up along a smooth center has the structure of a $\mathbf{P}^1$ fiber bundle, the projectification of the normal bundle of the blowing up center,

and

(iv). the restriction of the exceptional divisor to itself is equal to the first Chern class of the tautological line bundle, and its various self-intersections can be expressed by the Chern classes of the normal bundle (see page 47-51 of [F] or page 270 of [BT] for a corresponding statement in the cohomology ring).

By combining (i)-(iv)., we reduce the intersection numbers of $E_{i;j}$, $c_{total}(\mathbf{T}M_n)$, $c_{total}(f^*_{n-1,k}\mathbf{T}M_k)$, π_i^*C in $\mathcal{A}_0(M_n)$ to the intersection numbers of $\pi_i^*c_{total}(\mathbf{T}M)$ and π_i^*C, $1\leq i\leq n$ in $\mathcal{A}_0(M^n)$. As the only non-vanishing pairings among these classes can be expressed as polynomials in terms of $\pi_i^*C^2$, $\pi_i^*C\cap c_1(\mathbf{T}M)$, $\pi_i^*c_1(\mathbf{T}M)^2$ and $\pi_i^*c_2(\mathbf{T}M)$, $1\leq i\leq n$, the integral valued intersection number is a degree n homogeneous polynomial of $C^2\cap[M]$, $C\cap c_1(M)\cap M$, $c_1(M)^2\cap[M]$, $c_2(M)\cap[M]$. $\Box$

Remark 15 *When the irregularity $q=0$, $\mathcal{ASW}(C)=1$ because it is the top intersection pairing of $c_1(\mathbf{H})$ on a projective space $\mathbf{P}(\mathbf{V})$. When $q>0$, the $\mathcal{ASW}(C)=\sum_{a+2b=q,a,b\in\mathbf{N}\cup\{0\}}(-1)^a\int_{T(M)}\frac{ch_1^a}{a!}\cap\frac{ch_2^b}{b!}$, $ch_1=\pi_{T(M)*}(\frac{c_1(\mathcal{L}_{univ})^2}{2}\cap(2C+c_1(\mathbf{T}M)))$, $ch_2=\pi_{T(M)*}\frac{c_1(\mathcal{L}_{univ})^4}{4!}$, depends on the top intersection pairing on $T(M)$ and was calculated in [LL1], [LL2] in the topological category. Over here $\pi_{T(M)}:M\times T(M)\mapsto T(M)$ denote the Cartesian projection to $T(M)$.*

Remark 16 *In the above discussion, the mixed family invariant enumerates all curves within the family $M_{n+1} \times_{M_n} Y(\Gamma) \mapsto Y(\Gamma)$ and dual to $C - \mathbf{M}(E)E$. We do not require the image curve in M to lie within a particular complete linear system associated with a holomorphic line bundle over M. In case we restrict the holomorphic structure to a $t_L \in T(M)$, one may insert the zero cycle class $\{t_L\}$ into the family invariant. This has the effect of reducing the torus $T(M)$ to a single point and the modified mixed invariant associated to $Y(\Gamma)$ is of the form*

$$\mathcal{AFSW}^*_{M_{n+1}\times_{M_n}Y(\Gamma)\times\{t_L\}\mapsto Y(\Gamma)\times\{t_L\}}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$$

and the generic modified family invariant is of the form

$$\mathcal{AFSW}^*_{M_{n+1}\times\{t_L\}\mapsto M_n\times\{t_L\}}(1, C-\mathbf{M}(E)E).$$

An analogue of proposition 13 holds while we replace $\mathcal{ASW}(C)$ by $\mathcal{ASW}([t_L], C) = 1$.

5.3 The Combinatorics Involved in the Enumerations

In this subsection, we address the combinatorial issues regarding the linear ordering $\models$ and the partial ordering $\gg, \sqsupset$, involved in the blowing up construction and the inclusion relation on the various restricted family moduli spaces $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i} \times_{M_n} Y(\Gamma)$, $\Gamma \in \Delta(n)$. As will be demonstrated later, it has significant implications on the enumeration problem.

Let us start by noticing that,

Lemma 21 *The localized top Chern class contribution along D_Γ,*

$$\sum_{1\le i\le e}(-1)^{i-1}c_{e-i}(E\otimes_{\Gamma'\in I_\Gamma}\mathcal{O}(-D_{\Gamma'})|_{D_\Gamma})D_\Gamma^{i-1}[D_\Gamma]$$

can be identified with

$$\sum_{1\le i\le e}(-1)^{i-1}c_{e-i}(E\otimes_{\Gamma'\in I_\Gamma; Y(\Gamma)\cap Y(\Gamma')\neq\emptyset}\mathcal{O}(-D_{\Gamma'})|_{D_\Gamma})D_\Gamma^{i-1}[D_\Gamma].$$

I.e. in evaluating the localized contribution of the family invariant along D_Γ, one may remove those $\mathcal{O}(D_{\Gamma'})$ with $Y(\Gamma')\cap Y(\Gamma)=\emptyset$.

Proof: The defining section of $\mathcal{O}(D_{\Gamma'})$ vanishes exactly on $D_{\Gamma'}$. If $Y(\Gamma')\cap Y(\Gamma)=\emptyset$, $D_{\Gamma'}$ is totally disjoint from D_Γ in the space $\tilde{X}$. Thus, the line bundle $\mathcal{O}(D_{\Gamma'})|_{D_\Gamma}$ is isomorphic to the trivial bundle $\mathbf{C}|_{D_\Gamma}$. Therefore they can be removed from the expression involving Chern classes of $\mathcal{O}(D_{\Gamma'})|_{D_\Gamma}$. $\Box$

Because of lemma 21, in identifying the localized top Chern class contribution from D_Γ we may discard all the $Y(\Gamma')$, $\Gamma' \in I_\Gamma$, which do not intersect $Y(\Gamma)$ at all. Consider all the $Y(\Gamma')$, $\Gamma' \in I_\Gamma$, and digest briefly the geometric structure of $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma)$ relative to the various $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma')$ which touch it non-trivially.

Proposition 14 *Let e_i, and e'_i, $1 \leq i \leq n$ be the type I exceptional classes over Y_Γ and $Y_{\Gamma'}$, respectively. The restriction of the family moduli space $\mathcal{M}_{C-\mathbf{M}(E)E}$ to $Y(\Gamma)$, $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma) = Z(s_{canon}) \times_{M_n} Y(\Gamma)$, can be identified as the scheme theoretical union of the images of the primary component $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i;e_i\cdot(C-\mathbf{M}(E)E)<0} e_i} \times_{M_n} Y(\Gamma)$ and of the union of secondary components,*

$$\cup_{\Gamma' \in I_\Gamma} \mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_j;e'_j\cdot(C-\mathbf{M}(E)E)<0} e'_j} \times_{M_n} (Y(\Gamma) \cap Y(\Gamma'))$$

under the natural inclusions $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i;e_i\cdot(C-\mathbf{M}(E)E)<0} e_i} \subset \mathcal{M}_{C-\mathbf{M}(E)E}$ and $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e'_i;e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i} \subset \mathcal{M}_{C-\mathbf{M}(E)E}$, respectively.

Proof of proposition 14: Firstly we identify them on the set theoretical level. Let $z \in \mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma)$. The point z represents an algebraic curve dual to $C - \mathbf{M}(E)E$ above a point in $Y(\Gamma)$. Suppose that z is in the subspace $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} S_\Gamma$, then z is above a point $b \in S_\Gamma$ and $\mathcal{EC}_b(C - \mathbf{M}(E)E; Q) = \mathcal{C}_\Gamma$. Then e_i with $e_i \cdot (C - \mathbf{M}(E)E) < 0$ are exactly the generators of $\mathcal{C}_\Gamma$. This implies that the effective curve dual to $C - \mathbf{M}(E)E$ represented by z must contain irreducible components dual to each of the $e_i \in \mathcal{C}_\Gamma$. Thus, z is in the image of $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i;e_i\cdot(C-\mathbf{M}(E)E)<0} e_i} \subset \mathcal{M}_{C-\mathbf{M}(E)E}$.

If $z \in \mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} (Y(\Gamma) - S_\Gamma)$, then z is above a point [61] $b \in Y(\Gamma) - S_\Gamma \subset \cup_{\Gamma' \in \Delta(n); \Gamma' \prec \Gamma} S_{\Gamma'}$. In particular, $b \in Y_{\Gamma'}$ for some $\Gamma' \in \Delta(n)$. Let e'_i with $e'_i \cdot (C - \mathbf{M}(E)E) < 0$ be the type I exceptional classes over $Y_{\Gamma'}$ which generate the simplicial cone $\mathcal{C}_{\Gamma'}$. Then a similar argument applies as well and z is in the image of $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e'_i;e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i} \subset \mathcal{M}_{C-\mathbf{M}(E)E}$.

To identify them on the scheme theoretical level, simply realize that the difference of $Z(s_{canon}) \times_{M_n} Y(\Gamma)$ from $Z(s^\circ_{canon}) \times_{M_n} Y(\Gamma)$ can be analyzed by the various analogues of s°_{canon} and $\mathbf{W}^\circ_{canon}$, involving $\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i$ over $Y_{\Gamma'}$, $Y(\Gamma') \cap Y(\Gamma) \neq \emptyset$. By induction, we may get the equality on the sub-schemes of X. $\Box$

Remark 17 *In the special case when $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i) \gg (\Gamma', \sum_{e'_j\cdot(C-\mathbf{M}(E)E)<0} e'_j)$ (consult definition 11 for its definition), we have $Y(\Gamma') \subset Y(\Gamma)$ and $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e'_j\cdot(C-\mathbf{M}(E)E)<0} e'_j} \times_{M_n} Y(\Gamma')$ is naturally embedded into $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i} \times_{M_n} Y(\Gamma)$ as a sub-scheme[62]. Thus we may ignore such $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e'_j\cdot(C-\mathbf{M}(E)E)<0} e'_j} \times_{M_n}$*

[61] The *subset* is a consequence of lemma 13 and lemma 14.

[62] And therefore into $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma)$ automatically.

$Y(\Gamma')$ in the above union.

$\diamondsuit$ Heuristically we may interpret the blowing ups of (the strict transform of) $\mathcal{M}_{C-\mathbf{M}(E)E}\times_{M_n}Y(\Gamma')$ into $D_{\Gamma'}$ and factorizing s_{canon} by $D_{\Gamma'}$ as a mean of removing the contribution of the top Chern class of $\pi_X^*\mathbf{W}_{canon}\otimes\mathbf{H}$ from the image scheme of $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i}\times_{M_n}Y(\Gamma)$ inside $\mathcal{M}_{C-\mathbf{M}(E)E}\times_{M_n}Y(\Gamma)$. This explains why the localized contribution is expected to be related to some sort of mixed family invariant attached to $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i}$ over $Y(\Gamma)$, as had been demonstrated in theorem 4 of [Liu5].

It is vital to reflect and ask the following question,
$\diamondsuit$ **Question**: How to enumerate/identify the exact localized contribution of top Chern class along D_Γ? Do we expect the answer to be expressible as a mixed invariant or do we expect to get additional "correction terms"?

If there are additional correction terms, we have to understand where do these terms come from!

In fact when we blow up the various $D_{\Gamma'}$, $\Gamma'\in\Delta(n)$, $\Gamma\models\Gamma'$, we have also blown up along the image of $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e'_j\cdot(C-\mathbf{M}(E)E)<0}e'_j}\times_{M_n}Y(\Gamma')$ in $\mathcal{M}_{C-\mathbf{M}(E)E}\times_{M_n}Y(\Gamma)$, with $\Gamma\gg\Gamma'$ as well.

This suggests that we have also removed "accidentally" some additional contributions of the family invariant from $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot\mathbf{M}(E)E<0}e_i}\times_{M_n}Y(\Gamma)$ as well[63]. Therefore, we should expect to identify the localized contribution with a "modified" object instead of a normal mixed family invariant of $C-\mathbf{M}(E)E-\sum_{e_i\cdot\mathbf{M}(E)E<0}e_i$ over $Y(\Gamma)$.

This explains why in the definitions of the modified invariants, there are "correction terms" from $\Gamma'\ll\Gamma$ besides the dominated leading term. While these correction terms appear naturally in our current setting of the blowing up construction and the residual intersection theory, it is also needed to avoid the troublesome problem of over-subtracting (see the beginning of section 6.5 of [Liu5] for an explanation).

The complete answer to the above question will be given in section 6 where we identify the localized algebraic family invariant contribution along D_Γ with $\mathcal{AFSW}^*$ inductively. Before we present the proof, some additional knowledge about the geometric/combinatorial structure of $\mathcal{M}_{C-\mathbf{M}(E)E}$ is essential.

Consider the partial ordering $\sqsupset$,

Definition 15 *Let $\Gamma,\Gamma'\in\Delta(n)$. The pair $(\Gamma,\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i)$ is greater than $(\Gamma',\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i)$ under $\sqsupset$ if*

(i). $Y(\Gamma)\cap Y(\Gamma')\neq\emptyset$.

(ii). The combination of type I classes $\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i-\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i$ is semi-effective over $Y(\Gamma)\cap Y(\Gamma')$. I.e. the combination is either identically zero or is represented by effective curves over $Y(\Gamma)\cap Y(\Gamma')$.

[63]by the observation in remark 17.

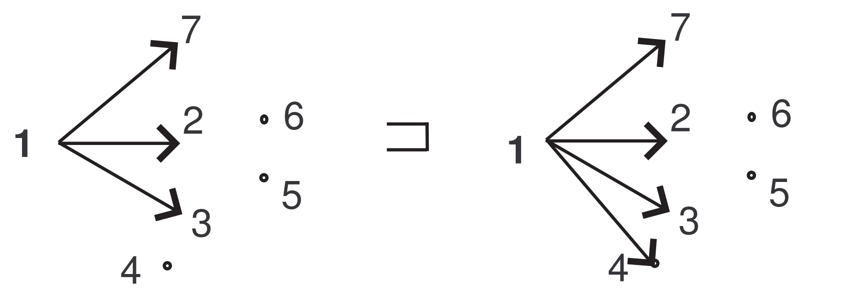

fig.6
a pair of admissible graphs in $adm(6)$ related by the $\sqsupset$ partial ordering.

Geometrically the partial ordering $\sqsupset$ signalizes that some of the type I curves dual to $e_i, e_i\cdot(C-\mathbf{M}(E)E)<0$, break into more than one component over $Y(\Gamma)\cap Y(\Gamma')$ and these $e'_j, e'_j\cdot(C-\mathbf{M}(E)E)<0$ are dual to certain components among them. When $\Gamma\sqsupset\Gamma'$, the semi-effectiveness of $\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i-\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i$ over $Y(\Gamma)\cap Y(\Gamma')$ implies that $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i}\times_{M_n}Y(\Gamma)\cap Y(\Gamma')$ is embedded into $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i}\times_{M_n}Y(\Gamma)\cap Y(\Gamma')$. In other words, the image of $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i}\times_{M_n}Y(\Gamma)$ in $\mathcal{M}_{C-\mathbf{M}(E)E}\times_{M_n}Y(\Gamma)$ does not capture $\mathcal{M}_{C-\mathbf{M}(E)E}$ above $Y(\Gamma)\cap Y(\Gamma')$ accurately and the two loci $Z(s^{\circ}_{canon})=\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i}$ and $Z(s_{canon})=\mathcal{M}_{C-\mathbf{M}(E)E}$ may differ over $Y(\Gamma)\cap Y(\Gamma')$.

Refer to fig.6 for an example[64].

Remark 18 *These two partial orderings $\gg$ and $\sqsupset$ are exclusive in the following sense that if $(\Gamma,\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i)\gg(\Gamma',\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i)$, then the expression $\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i-\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i$ is anti-effective over $Y(\Gamma)\cap Y(\Gamma')=Y(\Gamma')$. Thus, the $(\Gamma,\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i)$ cannot be greater than $(\Gamma',\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i)$ under $\sqsupset$.*

Consider the following setting among three admissible graphs. Fix a $\Gamma\in\Delta(n)$ and let $\Gamma_1\in I_\Gamma$ satisfy $Y(\Gamma)\cap Y(\Gamma_1)\neq\emptyset$. Let $\Gamma_2\in adm(n)$ with $\Gamma_2<\Gamma$, $\Gamma_2\leq\Gamma_1$, be an admissible graph. This implies that $Y_{\Gamma_2}\subset Y(\Gamma)\cap Y(\Gamma_1)$.

In the following proposition we discuss the few possibilities which can occur,

Proposition 15 *Suppose that $\Gamma_2\not\in\Delta(n)$, then $Y_{\Gamma_2}\in S_{\Gamma'}$ for some $\Gamma'\in\Delta(n)$. If $\Gamma_2\in\Delta(n)$, take $\Gamma'=\Gamma_2$ itself.*

As usual, let e'_i, $1\leq i\leq n$, denote the type I exceptional classes over $Y_{\Gamma'}$. Then either

[64] To simplify the notations, the $\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i$ parts have been skipped.

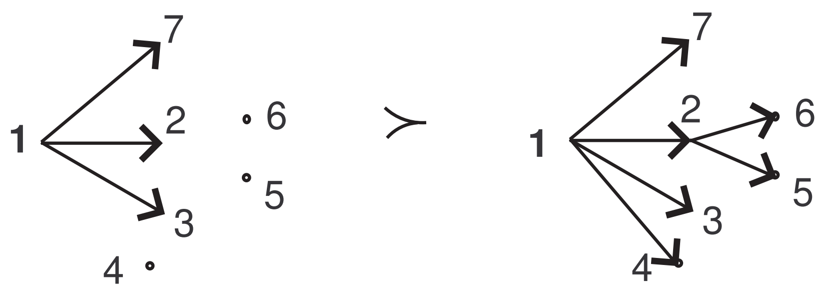

fig.7
a pair of admissible graphs in $adm(7)$ related by the $\succ$ partial ordering.

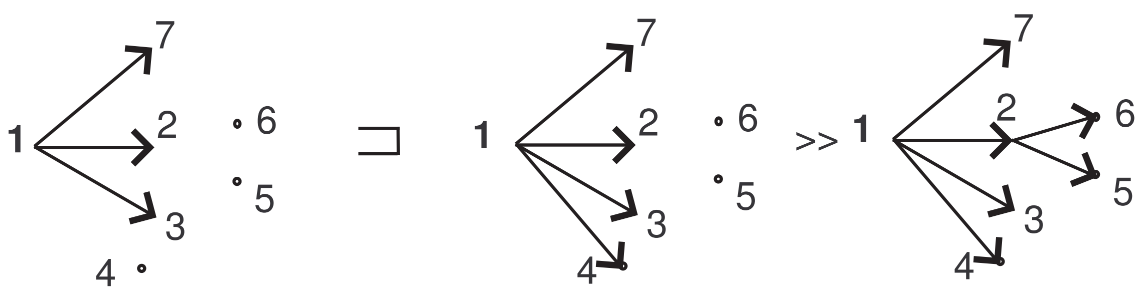

fig.8
following fig.7 above, the inserted admissible graph $\in adm(7)$ in the middle is $\sqsubset$ than the admissible graph on the left hand side, but $\gg$ than the admissible graph on the right hand side. this situation corresponds to **proposition** 15 case (b).

(a). $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i) \gg (\Gamma', \sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i)$.
or (b). $\exists$ *an intermediate* $\Gamma'' \in \Delta(n)$, $\Gamma'' \neq \Gamma$, *such that* $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i) \sqsupset (\Gamma'', \sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0} e''_i)$ *and* $(\Gamma'', \sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0} e''_i) \gg (\Gamma', \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e'_i)$.

Suppose that Γ' is as described in the statement of the proposition, then $Y(\Gamma) \cap S_{\Gamma'} \neq \emptyset$. It implies that $\Gamma \succ \Gamma'$ and then $\Gamma' \in I_\Gamma$.

The figure 7, 8 illustrate how proposition 15 holds in a special case.

Proof of proposition 15: If $\Gamma_2 \in \Delta(n)$, take $\Gamma' = \Gamma_2$. If $\Gamma_2 \notin \Delta(n)$, by proposition 11 and lemma 13, $Y_{\Gamma_2} \subset S_{\Gamma'}$ for some $\Gamma' \in \Delta(n)$. We know that Γ' cannot be Γ itself. Otherwise it implies immediately that $\mathcal{C}_\Gamma \supset \mathcal{C}_{\Gamma_1}$ because now we have $Y_{\Gamma_2} \subset S_{\Gamma'} = S_\Gamma$ (where the exceptional cone is equal to $\mathcal{C}_\Gamma$) and because $Y_{\Gamma_2} \subset Y(\Gamma_1)$. But this implies that $\Gamma_1 \succ \Gamma$ and then such a Γ_1 can **NOT** be in I_Γ. A contradiction to our assumption!

Therefore one may replace Γ_2 by some $\Gamma' \in \Delta(n)$. In any case we still have $Y(\Gamma') \cap Y(\Gamma) \supset Y(\Gamma_2) \cap Y(\Gamma) \neq \emptyset$.

Consider the cone $\mathcal{C}_{\Gamma'}$. Because $Y_{\Gamma_2} \subset S_{\Gamma'}$, the extremal generators [65] of $\mathcal{C}_{\Gamma'}$

[65] I.e. primitive generators of one-edges (extremal rays) at the boundary of the cone.

are exactly the type I exceptional classes over Y_{Γ_2} which pair negatively with $C - \mathbf{M}(E)E$. On the other hand, $Y_{\Gamma_2} \subset Y(\Gamma)$ (therefore $S_{\Gamma'} \cap Y(\Gamma) \neq \emptyset$ and this implies that $\mathcal{C}_\Gamma \subset \mathcal{C}_{\Gamma'}$.

We separate our discussion into a few cases.

(A). Suppose that all the extremal generators (among the type I exceptional classes) of the simplicial cone $\mathcal{C}_\Gamma$ remain to be the extremal generators of the simplicial cone $\mathcal{C}_{\Gamma'}$: As we know $\mathcal{C}_\Gamma \neq \mathcal{C}_{\Gamma'}$, there must be additional extremal generators of $\mathcal{C}_{\Gamma'}$ away from the boundary of $\mathcal{C}_\Gamma$. Then $Y(\Gamma)$ is the locus of co-existence of the type I classes e_i, $e_i \cdot (C - \mathbf{M}(E)E) < 0$. Then $Y(\Gamma')$ is characterized (by proposition 4) as the co-existence loci of all $e_i, e_i \cdot (C - \mathbf{M}(E)E) < 0$ and some other type I exceptional classes. So we know $Y(\Gamma') \subset Y(\Gamma)$ and we have,

$$(\Gamma, \sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i) \gg (\Gamma', \sum_{e'_i \cdot (C - \mathbf{M}(E)E) < 0} e'_i).$$

(B). Suppose that at least one of the extremal generators (among the type I exceptional classes) fails to be an extremal generator of $\mathcal{C}_{\Gamma'}$. Then the curve it represents must break into more than one irreducible component above $Y(\Gamma_2)$. Define an index set $P \subset \{1, 2, \cdots, n\}$ to consist of all the subscripts i of e_i such that the extremal generator e_i of the cone $\mathcal{C}_\Gamma$ fails to be an extremal generator of $\mathcal{C}_{\Gamma'}$.

By proposition 4 of [Liu4], each $e_i, i \in P$, can be expressed uniquely as an effective integral combination of the extremal generators of the simplicial cone $\mathcal{C}_{\Gamma'}$. Let the index set $P'' \subset \{1, 2, \cdots, n\}$ consists of all the subscripts j so that e'_j involve in the integral effective combination of at least one $e_i, i \in P$. Then we may write $e_i = \sum_{j \in P''} c_{i,j} e'_j$, with $c_j \geq 0$. We know that $\sum_{i \in P} c_{i,j} \geq 1$ for all $j \in P''$.

Then the collection of type I classes $e'_i, i \in P''$, generate a simplicial subcone $\mathcal{C}$ of $\mathcal{C}_{\Gamma'}$. Because $e'_i \cdot e'_j \geq 0$ for $i \neq j$ in P'', by proposition 3 of section 2, $\exists$ an admissible graph Γ'' such that $e''_i = e'_i$ for $i \in P''$ and e''_i are -1 classes for $i \notin P''$. The co-existence locus of $e'_i, i \in P''$ characterizes the closure of the admissible stratum $Y(\Gamma'')$ with $\mathcal{C}_{\Gamma''} = \mathcal{C}$ and since (by definition) $e''_j = e'_j, j \in P''$, they are exactly the type I exceptional classes above $Y_{\Gamma''}$ which pair negatively with $C - \mathbf{M}(E)E$.

Apparently by the construction of Γ'' we have[66],

$$(\Gamma'', \sum_{e''_i \cdot (C - \mathbf{M}(E)E) < 0} e''_i) \gg (\Gamma', \sum_{e'_i \cdot (C - \mathbf{M}(E)E) < 0} e'_i).$$

We may rewrite $\sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i - \sum_{e''_j \cdot (C - \mathbf{M}(E)E) < 0} e''_j$ as

$$\sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0; i \notin P} e_i + \sum_{i \in P} e_i - \sum_{j \in P''} e''_j = \sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0; i \notin P} e_i + \sum_{i \in P} \sum_{j \in P''} c_{i,j} e''_j - \sum_{j \in P''} e''_j$$

[66]Because non -1 classes e''_i are selected from e'_j.

$$= \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0; i\notin P} e_i + \{\sum_{j\in P''}\big((\sum_{i\in P} c_{i,j}) - 1\big)e''_j\}.$$

The first sum in the final expression is semi-effective over $Y(\Gamma)$ because it is a semi-effective combinations of some $e_i, i\notin P$ which are effective over $Y(\Gamma)$. On the other hand, the second sum in the final expression is semi-effective over $Y(\Gamma'')$ because (i). Each $e''_j, j\in P''$ is effective over $Y(\Gamma'')$ and P'' is defined to be the index set containing all the j such that $e''_j = e'_j$ are used in expressing $e_i, i\in P$.

So for any fixed $j\in P''$, the sum $\sum_{i\in P} c_{i,j}\geq 1$ and therefore $(\sum_{i\in P} c_{i,j}) - 1\geq 0$. So the sum of these two expressions is still semi-effective over the intersection $Y(\Gamma)\cap Y(\Gamma'')$. So the existence of such an intermediate Γ'' has been proved. $\Box$

As a consequence of proposition 15, the image of the moduli space $\mathcal{M}_{C-\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i}\times_{M_n} Y(\Gamma')$ into $\mathcal{M}_{C-\mathbf{M}(E)E}\times_{M_n} Y(\Gamma)$ is either contained directly in the image of $\mathcal{M}_{C-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i}\times_{M_n} Y(\Gamma)$ in $\mathcal{M}_{C-\mathbf{M}(E)E}\times_{M_n} Y(\Gamma)$ itself or is contained in the image of some intermediate $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0} e''_i}\times_{M_n} Y(\Gamma'')$. This gives an additional hierarchical structure among the various $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i}$.

When we attempt to identify the localized algebraic family invariant contribution to D_Γ, the above observation motivates us to reduce the list of blowing ups preceding Γ, I_Γ, by the following recipe:

Definition 16 *Define the subset $\bar{I}_\Gamma\subset I_\Gamma$ by removing all those $\Gamma'\in I_\Gamma$ such that (i). $S_{\Gamma'}\cap Y(\Gamma)=\emptyset$ or (ii). $\exists\Gamma''\in I_\Gamma$ with $(\Gamma'', \sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0} e''_i)\gg(\Gamma', \sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i)$.*

The reduced index set $\bar{I}_\Gamma$ collects all the $\Gamma'\in I_\Gamma$ which satisfy either $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)\gg(\Gamma', \sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i)$ or $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)\sqsupset(\Gamma', \sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i)$.

The set $\bar{I}_\Gamma$ inherits a linear ordering from I_Γ, still denoted by $\models$.

For all $\Gamma\in\Delta(n)$, we can define the corresponding $\bar{I}_\Gamma$ by the recipe of definition 16. The next lemma characterizes the relationship between the index set $\bar{I}_{\overline{\Gamma}}$ and $\bar{I}_\Gamma$ when $\overline{\Gamma}\ll\Gamma$.

Lemma 22 *Let $\Gamma\in\Delta(n)$ and let $\overline{\Gamma}\in\bar{I}_\Gamma$ satisfies $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)\gg(\overline{\Gamma}, \sum_{\overline{e}_i\cdot(C-\mathbf{M}(E)E)<0}\overline{e}_i)$. Then the reduced index set $\bar{I}_{\overline{\Gamma}}$ of $\overline{\Gamma}$ is the set of elements Γ' in $\bar{I}_\Gamma$ satisfying,*

(i). $Y(\overline{\Gamma})\cap S_{\Gamma'}\neq\emptyset$,

(ii). Γ' is smaller than $\overline{\Gamma}$ under the linear ordering [67] $\models$ in $\bar{I}_\Gamma$.

[67] Consult page 53 for its definition.

Proof: It is apparent that when $Y(\overline{\Gamma}) \subset Y(\Gamma)$, the condition $Y(\overline{\Gamma}) \cap S_{\Gamma'} \neq \emptyset$ (the direct consequence of $\Gamma' \in \bar{I}_{\overline{\Gamma}}$) implies $Y(\Gamma) \cap S_{\Gamma'} \neq \emptyset$. On the other hand $\overline{\Gamma} \ll \Gamma$ and $\Gamma'' \sqsubset \Gamma$ imply (see the next footnote below on page 72) $\overline{\Gamma} \sqsupset \Gamma''$. Thus to show that the condition (ii) is satisfied it suffices to show that for all $\Gamma' \in I_\Gamma - \bar{I}_\Gamma$, either it is due to

(1). Γ' is larger than or equal to $\overline{\Gamma}$ under the linear ordering $\models$, or

(2). $Y(\overline{\Gamma}) \cap S_{\Gamma'} = \emptyset$, or

(3). $\Gamma' \in I_{\overline{\Gamma}} - \bar{I}_{\overline{\Gamma}}$.

Suppose that $\Gamma' \in I_\Gamma - \bar{I}_\Gamma$, but the conditions (1). and (2). do not hold. We plan to show that (3). has to hold. Because of the violating of (1). an (2)., Γ' is smaller than $\overline{\Gamma}$ under the linear ordering $\models$ and $Y(\overline{\Gamma}) \cap S_{\Gamma'} \neq \emptyset$.

Then by the definition of $\bar{I}_\Gamma$, the assumption that Γ' is not in $\bar{I}_\Gamma$ and by proposition 15 (notice that Γ' satisfies the assumption of this proposition), there must exist some intermediate $\Gamma'' \in I_\Gamma \subset \Delta(n)$ such that

$$(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i) \sqsupset (\Gamma'', \sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0} e''_i) \gg (\Gamma', \sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i).$$

The condition $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i) \sqsupset (\Gamma'', \sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0} e''_i)$ is equivalent to the semi-effectiveness of $\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i - \sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0} e''_i$ over $Y(\Gamma'') \cap Y(\Gamma) \neq \emptyset$.

By our assumption on $\overline{\Gamma}$ in the statement of this lemma, $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i) \gg (\overline{\Gamma}, \sum_{\overline{e}_i\cdot(C-\mathbf{M}(E)E)<0} \overline{e}_i)$. This implies that for all the type I exceptional classes above Y_Γ satisfying $e_i\cdot(C-\mathbf{M}(E)E) < 0$, $\overline{e}_i = e_i$ is the corresponding type I exceptional class over $Y_{\overline{\Gamma}}$ and there are some additional $\overline{e}_s$ with $\overline{e}_s\cdot(C-\mathbf{M}(E)E) < 0$ other than those e_i, $e_i \cdot (C-\mathbf{M}(E)E) < 0$.

This implies that the class $\sum_{\overline{e}_j\cdot(C-\mathbf{M}(E)E)<0} \overline{e}_j - \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i$ has to be effective over $Y(\overline{\Gamma}) = \overline{Y(\overline{\Gamma})}$, which is a subset of $Y(\Gamma)$.

Thus,

$$\{\sum_{\overline{e}_j\cdot(C-\mathbf{M}(E)E)<0} \overline{e}_j - \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i\} + \{\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i - \sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0} e''_i\} = \sum_{\overline{e}_j\cdot(C-\mathbf{M}(E)E)<0} \overline{e}_j - \sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0} e''_i$$

is effective over the intersection of $Y(\Gamma'') \cap Y(\Gamma)$ and $Y(\overline{\Gamma})$, $Y(\Gamma'') \cap Y(\Gamma) \cap Y(\overline{\Gamma}) = Y(\Gamma'') \cap Y(\overline{\Gamma})$. In particular, the final expression is semi-effective over [68] $Y(\Gamma'') \cap Y(\overline{\Gamma})$ (since $Y(\overline{\Gamma}) \subset Y(\Gamma)$.

This implies that

$$(\overline{\Gamma}, \sum_{\overline{e}_i\cdot(C-\mathbf{M}(E)E)<0} \overline{e}_i) \sqsupset (\Gamma'', \sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0} e''_i) \gg (\Gamma', \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$$

as well (the $\gg$ inequality within the formula is already known).

[68] The argument essentially shows that $\Gamma \gg \overline{\Gamma}$, and $\Gamma \sqsupset \Gamma''$ imply $\overline{\Gamma} \sqsupset \Gamma''$.

So the element Γ' must be removed from $I_{\overline{\Gamma}}$ in forming $\bar{I}_{\overline{\Gamma}}$ and is not in $\bar{I}_{\overline{\Gamma}}$, either. $\Box$

Fixing a Γ'' such that $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i) \sqsupset (\Gamma'', \sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0} e''_i)$, the reduction from I_Γ to $\bar{I}_\Gamma$ enables us to group the family moduli spaces above $Y(\Gamma')$, $\mathcal{M}_{C-\mathbf{M}(E)E}\times_{M_n} Y(\Gamma')$, of all the Γ', satisfying $(\Gamma'', \sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0} e''_i) \gg (\Gamma', \sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i)$ together as sub-moduli spaces of $\mathcal{M}_{C-\mathbf{M}(E)E}\times_{M_n} Y(\Gamma'')$. Instead of blowing up all these $\mathcal{M}_{C-\mathbf{M}(E)E}\times_{M_n} Y(\Gamma')$ individually, we blow up the whole $\mathcal{M}_{C-\mathbf{M}(E)E}\times_{M_n} Y(\Gamma'')$ at once [69].

For all $\Gamma' \in \bar{I}_\Gamma$, it satisfies either $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i) \gg (\Gamma', \sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i)$, or $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i) \sqsupset (\Gamma', \sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i)$. For the enumeration purpose, we would like to rearrange the blowing up orderings (by $\models$) such that those Γ', with $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i) \gg (\Gamma', \sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i)$ are blown up later than those related to Γ by $\sqsupset$.

To achieve this goal, introduce a new linear ordering among $\bar{I}_\Gamma$, denoted by $\vdash$,

Definition 17 *Let $\Gamma_1 \in \bar{I}_\Gamma$. Define the "accumulation" $A_{\Gamma_1} = \{\Gamma_1\}$ if $\Gamma_1 \ll \Gamma$. Define $A_{\Gamma_1} = \{\Gamma' | \Gamma' \ll \Gamma_1, \Gamma' \in I_\Gamma\}$ if $\Gamma_1 \sqsubset \Gamma$. For all $\Gamma_1 \in \bar{I}_\Gamma$, each accumulation A_{Γ_1} has a unique smallest element under $\models$.*

Define a new linear ordering $\vdash$ on $\bar{I}_\Gamma$ by the following recipe:

(i). Suppose that both Γ_1 and Γ_2 are simultaneously $\sqsubset \Gamma$ or $\ll \Gamma$, define Γ_1 is greater than Γ_2 under $\vdash$ if the smallest element within the accumulation A_{Γ_1} under $\models$ is larger than (under $\models$) the smallest element of A_{Γ_2}.

(ii). Suppose that $\Gamma_1 \ll \Gamma$ but $\Gamma_2 \sqsubset \Gamma$, define Γ_1 is larger than Γ_2 under $\vdash$, i.e. $\Gamma_1 \vdash \Gamma_2$.

The heuristic motivation for such a new linear ordering is that the precedence among the sequence of blowing ups should be determined by the corresponding precedence of the smallest graph under $\models$ in the set A_{Γ_1}.

In the new linear ordering $\vdash$, those Γ' with $\Gamma' \ll \Gamma$ accumulate at the larger end of $\bar{I}_\Gamma$.

Definition 18 *Define $\bar{I}_\Gamma^{\gg} \subset \bar{I}_\Gamma$ to be the set of all elements Γ' in $\bar{I}_\Gamma$ such that the following condition $(\Gamma, \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i) \gg (\Gamma', \sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i)$ holds.*

Then definition 17 implies that an arbitrary element in $\bar{I}_\Gamma^{\gg}$ is greater than (under the newly defined $\vdash$) an arbitrary element in $\bar{I}_\Gamma - \bar{I}_\Gamma^{\gg}$.

The revised linear ordering $\vdash$ will play an essential role in the enumeration of the algebraic family Seiberg-Witten invariants in the next section.

[69] In section 6.1, proposition 16 implies that the re-grouping of the restricted family moduli spaces like this does not affect the localized contribution of the family invariant along D_Γ, thanks to the birational invariance of Segre classes of normal cones.

6 The Inductive Proof and the Identification with the Modified Family Invariants

The goal of this section is to finish up the proof of the main theorem by identifying the localized contribution of the original algebraic family Seiberg-Witten invariant of $C - \mathbf{M}(E)E$ over D_Γ with the modified algebraic family Seiberg-Witten invariant defined in subsection 5.2.

In subsection 5.3, we have introduced the reduced index sets $\bar{I}_\Gamma$ (see definition 16), its subset $\bar{I}_\Gamma^{\gg}$ (see definition 18) and the new linear ordering $\vdash$ (see definition 17). We may modify the original blowing up sequence by blowing ups the (strict transforms) of all the $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} \times Y(\Gamma') \subset \mathbf{P}(\mathbf{V}_{canon})$ for $\Gamma' \in \bar{I}_\Gamma$ instead of I_Γ, starting from the smallest element under $\models$ or $\vdash$ and along the reversed $\models$ or $\vdash$ orderings.

In section 6.1, we prove the independence of the localized top Chern class contribution along D_Γ to the detailed history of the blowing ups performed ahead of Γ under the reversed ordering of $\models$ or $\vdash$.

In section 6.2, we present the key argument to identify the integral of the cap product of top power of $c_1(\mathbf{H})$ with the localized contribution of top Chern class along D_Γ with the modified algebraic family Seiberg-Witten invariants.

Then in section 6.3 we finish the proof of the main theorem in the paper by combining the discussion in section 3, 4, 5 and the current section.

In section 6.4, we show with the help of Gööttsche's argument that $\mathcal{AFSW}^*_{M_{\delta+1}\times\{t_L\}\mapsto M_\delta\times\{t_L\}}(1, C-2\sum E_i)$ can be realized as a counting of discrete number of nodal curves in a generic δ-dimensional linear sub-system of a $5\delta-1$ very ample line bundle $L \mapsto M$.

6.1 The Independence of the Localized Top Chern Class Contribution to the Orderings of the Blowing Ups

It makes sense to ask the following question:

Question: For a fixed $\Gamma \in \Delta(n) - \{\gamma_n\}$, is the localized top Chern class contribution over D_Γ independent [70] to the "history" of the sequences of blowing ups we had performed on X before the one associated with Γ? Different choices of earlier blowing ups leads to mutually birational divisor D_Γ. Thus it becomes a non-trivial question to ask. More precisely suppose that we blow up the scheme $X = \mathbf{P}(\mathbf{V}_{canon})$ along the strict transformations of $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma')$ following the reversed ordering $(I_\Gamma, \models)$, $(\bar{I}_\Gamma, \models)$ or $(\bar{I}_\Gamma, \vdash)$, do we get "identical" localized top Chern class contributions upon the resulting exceptional divisor over $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma)$ in the three different cases?

The answer to this question is affirmative as will be shown in the proof of the following proposition.

[70]By 'independence', we do not mean for arbitrary blowing ups. See below for more details.

Before we state our result, let us introduce some notations.

Denote the consecutively blown ups of $X = \mathbf{P}(\mathbf{V}_{canon})$ along the strict transformations of $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma')$, $\Gamma' \in I_\Gamma$ following the reversed ordering of $\models$ by X_Γ. The $\pi_{X_\Gamma} : X_\Gamma \mapsto M_n \times T(M)$ is the projection map to $M_n \times T(M)$. Some repeated applications of the residual intersection formula of top Chern classes, proposition 8, results in the residual obstruction vector bundle $\pi^*_{X_\Gamma}\mathbf{W}_{canon} \otimes \mathbf{H} \otimes_{\Gamma' \in I_\Gamma} \mathcal{O}(-D_{\Gamma'})$.

Likewise we denote the consecutively blown ups of X along the strict transformations of $Z(s_{canon}) \times_{M_n} Y(\Gamma') = \mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma')$, $\Gamma' \in \bar{I}_\Gamma$ with the reversed ordering of $\models$ (or $\vdash$) by $\bar{X}_\Gamma$ with the projection map $\pi_{\bar{X}_\Gamma} : \bar{X}_\Gamma \mapsto M_n \times T(M)$ (or $\hat{X}_\Gamma$ with the projection map $\pi_{\hat{X}_\Gamma} : \hat{X}_\Gamma \mapsto M_n \times T(M)$). We denote the corresponding exceptional divisors by $\bar{D}_{\Gamma'}$ (or $\hat{D}_{\Gamma'}$), $\Gamma' \in \bar{I}_\Gamma$ and the corresponding residual obstruction vector bundle is $\pi^*_{\bar{X}_\Gamma}\mathbf{W}_{canon} \otimes \mathbf{H} \otimes_{\Gamma' \in \bar{I}_\Gamma} \mathcal{O}(-\bar{D}_{\Gamma'})$ (or $\pi^*_{\hat{X}_\Gamma}\mathbf{W}_{canon} \otimes \mathbf{H} \otimes_{\Gamma' \in \bar{I}_\Gamma} \mathcal{O}(-\hat{D}_{\Gamma'})$), respectively.

By blowing up the strict transform of $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma)$ in X_Γ, $\bar{X}_\Gamma$ or $\hat{X}_\Gamma$, denote the blown up schemes by $\tilde{X}_\Gamma, \tilde{\bar{X}}_\Gamma$ and $\tilde{\hat{X}}_\Gamma$, respectively. We denote the blown up exceptional divisors from the strict transformations of $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma) = Z(s_{canon}) \times_{M_n} Y(\Gamma)$ in X_Γ, $\bar{X}_\Gamma$, $\hat{X}_\Gamma$ by D_Γ, $\bar{D}_\Gamma$, $\hat{D}_\Gamma$, respectively.

By applying lemma 11, we get the corresponding localized contributions of the top Chern classes for $\pi^*_{X_\Gamma}\mathcal{W}_{canon} \otimes \mathbf{H} \otimes_{\Gamma' \in I_\Gamma} \mathcal{O}(-D_{\Gamma'})$, or for $\pi^*_{\bar{X}_\Gamma}\mathcal{W}_{canon} \otimes \mathbf{H} \otimes_{\Gamma' \in \bar{I}_\Gamma} \mathcal{O}(-\bar{D}_{\Gamma'})$ or for $\pi^*_{\hat{X}_\Gamma}\mathcal{W}_{canon} \otimes \mathbf{H} \otimes_{\Gamma' \in \bar{I}_\Gamma} \mathcal{O}(-\hat{D}_{\Gamma'})$ along D_Γ, $\bar{D}_\Gamma$ and $\hat{D}_\Gamma$, respectively.

These three spaces $X_\Gamma, \bar{X}_\Gamma, \hat{X}_\Gamma$ are all birational and all of them map onto X through the blowing down projection maps. The following proposition asserts that the images of the localized contributions of top Chern classes along D_Γ, $\bar{D}_\Gamma$ and $\hat{D}_\Gamma$ in $\mathcal{A}.(X)$ are all equal and provide an explanation.

Proposition 16 *The images in $\mathcal{A}_{dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{W}_{canon}}(X)$ of the localized contribution of the top Chern class for $\pi^*_X\mathcal{W}_{canon} \otimes \mathbf{H} \otimes_{\Gamma' \in I_\Gamma} \mathcal{O}(-D_{\Gamma'})$ over the strict transformation* [71] *of $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma)$ in X_Γ, the localized contribution of top Chern class for $\pi^*_{\bar{X}_\Gamma}\mathcal{W}_{canon} \otimes \mathbf{H} \otimes_{\Gamma' \in \bar{I}_\Gamma} \mathcal{O}(-\bar{D}_{\Gamma'})$ over the strict transform of $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma)$ in $\bar{X}_\Gamma$, and the localized contribution of top Chern class for $\pi^*_{\hat{X}_\Gamma}\mathcal{W}_{canon} \otimes \mathbf{H} \otimes_{\Gamma' \in \bar{I}_\Gamma} \mathcal{O}(-\hat{D}_{\Gamma'})$ over the strict transform of $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma)$ in $\hat{X}_\Gamma$ are all equal to each other.*

Proof: The key issue is to understand why the re-grouping of elements related by $\gg$ and the changing of the orderings from $\models$ to $\vdash$ do not affect the image cycle class of the above localized contributions of top Chern classes.

By using the residual intersection formula of the top Chern class, i.e. proposition 8, it implies that the images in $\mathcal{A}.(X)$ of the three localized contributions

[71] Along the blown up divisor $D_\Gamma \subset \tilde{X}_\Gamma$ above the strict transformation of $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma)$ in X_Γ.

of the top Chern classes along D_Γ, $\bar{D}_\Gamma$, and $\hat{D}_\Gamma$ can be identified with the push-forwards into $\mathcal{A}.(X)$ of

$$c_{rank_{\mathbf{C}}\mathbf{W}_{canon}}(\pi^*_{\tilde{X}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}\otimes_{\Gamma'\in I_\Gamma}\mathcal{O}(-D_{\Gamma'}))-c_{rank_{\mathbf{C}}\mathbf{W}_{canon}}(\pi^*_{\tilde{X}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}(-D_\Gamma)\otimes_{\Gamma'\in I_\Gamma}\mathcal{O}(-D_{\Gamma'})),$$

$$c_{rank_{\mathbf{C}}\mathbf{W}_{canon}}(\pi^*_{\tilde{\bar{X}}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}\otimes_{\Gamma'\in \bar{I}_\Gamma}\mathcal{O}(-\bar{D}_{\Gamma'}))-c_{rank_{\mathbf{C}}\mathbf{W}_{canon}}(\pi^*_{\tilde{\bar{X}}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}(-\bar{D}_\Gamma)\otimes_{\Gamma'\in \bar{I}_\Gamma}\mathcal{O}(-\bar{D}_{\Gamma'})),$$

and

$$c_{rank_{\mathbf{C}}\mathbf{W}_{canon}}(\pi^*_{\tilde{\hat{X}}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}\otimes_{\Gamma'\in \bar{I}_\Gamma}\mathcal{O}(-\hat{D}_{\Gamma'}))-c_{rank_{\mathbf{C}}\mathbf{W}_{canon}}(\pi^*_{\tilde{\hat{X}}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}(-\hat{D}_\Gamma)\otimes_{\Gamma'\in \bar{I}_\Gamma}\mathcal{O}(-\hat{D}_{\Gamma'})),$$

respectively.

On the other hand, because D_Γ, $\bar{D}_\Gamma$ and $\hat{D}_\Gamma$ map into $X\times_{M_n}Y(\Gamma)$ under the projections, so the images of the localized top Chern classes in $\mathcal{A}.(X)$ factor through the map $i_{\Gamma *}:\mathcal{A}.(X\times_{M_n}Y(\Gamma))\mapsto \mathcal{A}.(X)$.

On the other hand, $j:X-X\times_{M_n}Y(\Gamma)\subset X$ is open in X and by proposition 1.8. on page 21 of [F] we have the following exact sequence,

$$\mathcal{A}.(X\times_{M_n}Y(\Gamma))\xrightarrow{i_{\Gamma *}}\mathcal{A}.(X)\xrightarrow{j^*}\mathcal{A}.(X-X\times_{M_n}Y(\Gamma))\mapsto 0.$$

For any cycle β in $X-X\times_{M_n}Y(\Gamma)$, its Zariski closure $\bar{\beta}$ in X defines a cycle in X. This extension of cycles defines the right inverse of j^*. Thus for each $\alpha\in\mathcal{A}.(X)$, there exists a unique [72] cycle class $\alpha|_{X\times_{M_n}Y(\Gamma)}\in Im(i_{\Gamma *})$ such that $\alpha-\alpha|_{X\times_{M_n}Y(\Gamma)}=\overline{j^*\alpha}$. From now on we refer to $\alpha|_{X\times_{M_n}Y(\Gamma)}$ informally as the component of α inside the subspace $X\times_{M_n}Y(\Gamma)$.

Because the $\mathcal{A}.(X)$ images of the three intersection pairings are in $Im(i_{\Gamma *})$, it suffices to consider the push-forwards of the three top Chern classes in the first group, $c_{top}(\pi^*_{\tilde{X}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}\otimes_{\Gamma'\in I_\Gamma}\mathcal{O}(-D_{\Gamma'}))$, $c_{top}(\pi^*_{\tilde{\bar{X}}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}\otimes_{\Gamma'\in \bar{I}_\Gamma}\mathcal{O}(-\bar{D}_{\Gamma'}))$, and $c_{top}(\pi^*_{\tilde{\hat{X}}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}\otimes_{\Gamma'\in \bar{I}_\Gamma}\mathcal{O}(-\hat{D}_{\Gamma'}))$, and identify their components [73] inside $X\times_{M_n}Y(\Gamma)$. Then consider the push-forwards of the second group of three top Chern classes, $c_{top}(\pi^*_{\tilde{X}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}(-D_\Gamma)\otimes_{\Gamma'\in I_\Gamma}\mathcal{O}(-D_{\Gamma'}))$, $c_{top}(\pi^*_{\tilde{\bar{X}}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}(-\bar{D}_\Gamma)\otimes_{\Gamma'\in \bar{I}_\Gamma}\mathcal{O}(-\bar{D}_{\Gamma'}))$, and $c_{top}(\pi^*_{\tilde{\hat{X}}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}(-\hat{D}_\Gamma)\otimes_{\Gamma'\in \bar{I}_\Gamma}\mathcal{O}(-\hat{D}_{\Gamma'}))$ in $\mathcal{A}.(X\times_{M_n}Y(\Gamma)$, and identify their components inside $X\times_{M_n}Y(\Gamma)$.

$\diamondsuit$ Case I: We identify the $\mathcal{A}.(X\times_{M_n}Y(\Gamma))$-components of the push-forwards of the top Chern classes within the first group. The identification of the second group is similar, and will be handled in Case II below. Introduce three divisors [74] $D_1=(\cup_{\Gamma'\in I_\Gamma}D_{\Gamma'})\subset X_1=X_\Gamma$, $D_2=(\cup_{\Gamma'\in \bar{I}_\Gamma}\bar{D}_{\Gamma'})\subset X_2=\bar{X}_\Gamma$ and $D_3=$

[72] As $i_{\Gamma *}$ may not be injective, the uniqueness of the class in $\mathcal{A}.(X\times_{M_n}Y(\Gamma))$ is not ensured. Nevertheless its image in $\mathcal{A}.(X)$ is.

[73] Over here we do not intend to claim that the push-forwards of these top Chern classes are all equal in $\mathcal{A}.(X)$. In fact their j^*-restrictions in $\mathcal{A}.(X-X\times_{M_n}Y(\Gamma))$ may be different. The object we really care about is the differences of the push-forwards of top Chern classes and the cycle classes extended from $\mathcal{A}.(X-X\times_{M_n}Y(\Gamma)$ have to cancel out completely.

[74] We introduce the new notations X_a, D_a to avoid writing parallel formulae repeatedly!

$(\cup_{\Gamma' \in \bar{I}_\Gamma} \hat{D}_{\Gamma'}) \subset X_3 = \hat{X}_\Gamma$. The restriction of the vectors bundles involved in the first group are pull-back from X_1, X_2, and X_3, respectively. We define $\pi_{X_a} : X_a \mapsto M_n \times T(M)$. The yet-to-be-identified classes are $c_{top}(\pi_{X_i}^* \mathbf{W}_{canon} \otimes \mathbf{H} \otimes \mathcal{O}(-D_a))$ for $a = 1, 2, 3$. By applying the residual intersection formula, i.e. proposition 8, to the D_1, D_2 and D_3, these top Chern classes can be re-written as

$$c_{top}(\pi_{X_a}^* \mathbf{W}_{canon} \otimes \mathbf{H}) \cap [X_a] - \sum_{1 \le i \le rank_{\mathbf{C}} \mathbf{W}_{canon}} (-1)^{i-1} c_{rank_{\mathbf{C}} \mathbf{W}_{canon} - i}(\pi_{X_a}^* \mathbf{W}_{canon} \otimes \mathbf{H})(D_a)^{i-1}[D_a],$$

for $a = 1, 2, 3$.

The image of the first term into $\mathcal{A}.(X)$ is apparently independent of a and has a unique component in $X \times_{M_n} Y(\Gamma)$. So it suffices to show that the components in $\mathcal{A}.(X \times_{M_n} Y(\Gamma))$ of the push-forwards of the second terms involving D_a are a-independent.

Firstly recall that when D_a is a divisor, the total Chern class $c_{total}(\mathcal{O}(D_a)) = 1 + D_a$ and the total Segre class $s_{total}(\mathcal{O}(D_a)) = 1 + \sum_{j \ge 1} (-1)^j D_a^j$ is the total Segre class of the normal cone $s_{total}(D_a, X_a)$. So for $a = 1, 2, 3$, the push-forwards of the second terms can be re-expressed as the push-forward of

$$\eta_a = \{c_{total}(\big(\pi_{X_a}^* \mathbf{W}_{canon} \otimes \mathbf{H}\big)|_{D_a}) \cap s_{total}(D_a, X_a)\}_{dim_{\mathbf{C}} X - rank_{\mathbf{C}} \mathbf{W}_{canon}}$$

into $\mathcal{A}.(X)$.

Define $h_a : X_a \mapsto X$ to be the blowing down projection maps for $a = 1, 2, 3$.

Now recall the following proposition 4.2.(a) on page 74 of [F].

Proposition 17 *(Fulton) Let $f : Y' \mapsto Y$ be a morphism of pure-dimensional scheme, $Z \subset Y$ a closed sub-scheme, $Z' = f^{-1}(Z)$ the inverse image, $g' : Z' \mapsto Z$ the induced morphism.*

Suppose that f is proper, Y irreducible and f maps each irreducible component of Y' onto Y, then

$$g_*(s(Z', Y')) = deg(Y'/Y) \cdot s(Z, Y).$$

In our context, the blowing down map $h_a : X_a \mapsto X$ is proper, and $deg(X_a/X) = 1$ (because they are birational). The sub-scheme $Z = h_a(D_a) \times_{M_n} Y(\Gamma)$. Because h_a are composite blowing down maps, $D_a \times_{M_n} Y(\Gamma) = h_a^{-1}(h_a(D_a) \times_{M_n} Y(\Gamma))$. Both of X_a and X are irreducible and h_a maps X_a onto X.

By applying proposition 17, for all $1 \le a \le 3$ the push-forward-images of the above classes η_a in $\mathcal{A}.(h_a(D_a))$ are equal to

$$\{c_{total}(\big(\pi_X^* \mathbf{W}_{canon} \otimes \mathbf{H}\big)|_{h_a(D_a)}) \cap s(h_a(D_a), X)\}_{dim_{\mathbf{C}} X - rank_{\mathbf{C}} \mathbf{W}_{canon}}.$$

Then the fact that the $Im(i_{\Gamma *})$ components of their push-forwards into $\mathcal{A}.(X)$ are equal follows from the following observation,

Lemma 23 *Let $h_a : X_a \mapsto X$ and D_a, $1 \leq a \leq 3$ be as described above. Let the family moduli space of $C - \mathbf{M}(E)E$, $\mathcal{M}_{C-\mathbf{M}(E)E} \subset X$, denote the sub-scheme defined by the canonical section s_{canon} of $\pi_X^* \mathbf{W}_{canon} \otimes \mathbf{H}$. Then for $a = 1, 2, 3$, the sub-schemes $h_a(D_a) \cap \big(X \times_{M_n} Y(\Gamma)\big) = h_a(D_a) \times_{M_n} Y(\Gamma) \subset X$ all coincide and are all equal to the finite union $\cup_{\Gamma' \in I_\Gamma} \mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} \big(Y(\Gamma') \cap Y(\Gamma)\big)$.*

Proof: It is easy to see that the change of the linear ordering from $\models$ to $\vdash$ in $\bar{I}_\Gamma$ does not affect the total locus which is blown up. Thus we know that $h_2(D_2) = h_3(D_3) = \cup_{\Gamma' \in \bar{I}_\Gamma} \mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma')$. So their intersections with $X \times_{M_n} Y(\Gamma)$ are equal.

On the other hand, to argue that $h_1(D_1) \times_{M_n} Y(\Gamma) = h_2(D_2) \times_{M_n} Y(\Gamma) = \cup_{\Gamma' \in \bar{I}_\Gamma} \mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} \big(Y(\Gamma') \cap Y(\Gamma)\big)$, it suffices to show that for all $\Gamma' \in I_\Gamma - \bar{I}_\Gamma$, the corresponding sub-scheme $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} \big(Y(\Gamma') \cap Y(\Gamma)\big)$ has been included in the union of closed sub-schemes $\cup_{\Gamma'' \in \bar{I}_\Gamma} \mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} \big(Y(\Gamma'') \cap Y(\Gamma)\big)$ already.

We may assume that $Y(\Gamma') \cap Y(\Gamma) \neq \emptyset$ or the statement is trivial to prove. By lemma 15, we know that there are three exclusive possibilities (a). $\Gamma' \succ \Gamma$, (b). $\Gamma \succ \Gamma'$, (c). $\exists \Gamma'' \in \Delta(n)$ such that both of $\Gamma, \Gamma' \succ \Gamma''$.

Suppose that $S_{\Gamma'} \cap Y(\Gamma) = \emptyset$. We argue that we may replace Γ' by some Γ'' with $S_{\Gamma''} \cap Y(\Gamma) \neq \emptyset$. We already have $S_{\Gamma'} \cap Y(\Gamma) = \emptyset$ by our assumption. We know that $Y(\Gamma') \cap S_\Gamma = \emptyset$, too. If not, the hypothesis $Y(\Gamma') \cap S_\Gamma \neq \emptyset$ and lemma 14 imply $\Gamma' \succ \Gamma$ and therefore $\Gamma' \models \Gamma$. Then such a Γ' cannot be in the index set I_Γ at all. As both (a). and (b). fail, it falls into the situation (c). that $S_\Gamma \cap Y(\Gamma') = S_{\Gamma'} \cap Y(\Gamma) = \emptyset$. From the proof of lemma 15 we know that for "all" $b \in Y(\Gamma) \cap Y(\Gamma')$, there exists a $\Gamma'' \in \Delta(n)$ such that $b \in S_{\Gamma''} \cap Y(\Gamma) \neq \emptyset$.

From this digestion we learn that $\cup_{\Gamma' \in I_\Gamma - \bar{I}_\Gamma} \mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} \big(Y(\Gamma') \cap Y(\Gamma)\big)$ can be replaced by the union $\cup_{\Gamma' \in I_\Gamma; S_{\Gamma'} \cap Y(\Gamma) \neq \emptyset} \mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} \big(S_{\Gamma'} \cap Y(\Gamma)\big)$.

According to definition 16 on the reduced index set $\bar{I}_\Gamma$, any $\Gamma' \in I_\Gamma$ with $S_{\Gamma'} \cap Y(\Gamma) \neq \emptyset$ is thrown away to form $\bar{I}_\Gamma$ exactly when there exists another $\Gamma'' \in \bar{I}_\Gamma$ with $(\Gamma'', \sum_{e''_i \cdot (C-\mathbf{M}(E)E)<0} e''_i) \gg (\Gamma', \sum_{e'_i \cdot (C-\mathbf{M}(E)E)<0} e'_i)$ and $(\Gamma, \sum_{e_i \cdot (C-\mathbf{M}(E)E)<0} e_i) \sqsupset (\Gamma'', \sum_{e''_i \cdot (C-\mathbf{M}(E)E)<0} e''_i)$.

However the $\gg$ relationship (see definition 11) between Γ'' and Γ' implies $Y(\Gamma'') \supset Y(\Gamma')$. This implies that $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} S_{\Gamma'} \subset \mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma'')$. So we have the inclusion

$$\cup_{\Gamma' \in I_\Gamma; S_{\Gamma'} \cap Y(\Gamma) \neq \emptyset} \mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} \big(S_{\Gamma'} \cap Y(\Gamma)\big) \subset \cup_{\Gamma'' \in \bar{I}_\Gamma} \mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} \big(Y(\Gamma'') \cap Y(\Gamma)\big).$$

By combining the inclusions we know that $h_1(D_1) \times_{M_n} Y(\Gamma)$ must be included in $h_2(D_2) \times_{M_n} Y(\Gamma)$. But the reversed inclusion $h_2(D_2) \times_{M_n} Y(\Gamma) \subset h_1(D_1) \times_{M_n} Y(\Gamma)$ is apparent. So we have $h_1(D_1) \times_{M_n} Y(\Gamma) = h_2(D_2) \times_{M_n} Y(\Gamma) = h_3(D_3) \times_{M_n} Y(\Gamma)$ as sub-schemes of X and the lemma is proved. $\Box$

As usual let $i_{h_a(D_a)} : h_a(D_a) \hookrightarrow X$ be the inclusion maps. The above lemma tells us that the restriction of $h_a(D_a)$ to $X \times_{M_n} Y(\Gamma)$ coincide. On the other hand the normal cones $\mathbf{C}_{h_a(D_a)} X$ can always be written as the unions of irreducible normal cones supporting over irreducible components of the sub-schemes

$h_a(D_a)$. By separating the irreducible components of $h_a(D_a)$ in $X \times_{M_n} Y(\Gamma)$ and the zariski-closures of $h_a(D_a) \cap (X - X \times_{M_n} Y(\Gamma))$, we may write each $\mathbf{C}_{h_a(D_a)}X = \mathbf{C}_{h_a(D_a)\times_{M_n} Y(\Gamma)}X \cup \mathbf{C}'_a$, with $\mathbf{C}'_a$ supported over the zariski-closure of $h_a(D_a) \cap (X - X \times_{M_n} Y(\Gamma)) = h_a(D_a) \times_{M_n} (M_n - Y(\Gamma))$ in X. Then it is easy to see that the components along $X \times_{M_n} Y(\Gamma)$ of $i_{h_a(D_a)*}\{s(\mathbf{C}_{h_a(D_a)}X)\} = i_{h_a(D_a)*}\{s(\mathbf{C}_{h_a(D_a)\times_{M_n} Y(\Gamma)}X)\} + i_{h_a(D_a)*}\{s(\mathbf{C}'_a)\}$ is exactly $i_{h_a(D_a)*}\{s(\mathbf{C}_{h_a(D_a)\times_{M_n} Y(\Gamma)}X)\}$ while $i_{h_a(D_a)*}\{s(\mathbf{C}'_a)\}$ are the extension $\mathcal{A}.(X - X \times_{M_n} Y(\Gamma)) \mapsto \mathcal{A}.(X))$ of the restricted Segre class $i_{h_a(D_a)*}\{s(h_a(D_a) \times_{M_n} (M_n - Y(\Gamma)), X - X \times_{M_n} Y(\Gamma))\}$.

By the above lemma, we know that the $X \times_{M_n} Y(\Gamma)$-components of the total Segre classes $i_{h_a(D_a)*}s(h_a(D_a), X)$ coincide for all $1 \leq a \leq 3$. By capping with $c_{total}(\pi_X^* \mathbf{W}_{canon} \otimes \mathbf{H})$, we conclude that the components in $X \times_{M_n} Y(\Gamma)$ of $i_{h_a(D_a)*}\{c_{total}\big((\pi_X^*\mathbf{W}_{canon} \otimes \mathbf{H})\big) \cap s(h_a(D_a), X)\}_{dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{W}_{canon}}$ are all the same. So Case I is proved.

$\diamondsuit$ Case II: The identifications of the components in $X \times_{M_n} Y(\Gamma)$ of the second groups of three top Chern classes are rather parallel to the previous argument in Case I, with some minute difference. We define $X'_1 = \tilde{X}_\Gamma$, $X'_2 = \tilde{\bar{X}}_\Gamma$ and $X'_3 = \tilde{\hat{X}}_\Gamma$. Define $h'_a : X'_a \mapsto X$ to be the projection maps.

Then the total transformations of $D_a \subset X_a$ under the pull-backs of the blowing down maps $X'_a \mapsto X_a$ define Cartier divisors in X'_a and we skip the pull-back notations and denote them by the same symbols [75] D_a. Set $D'_1 = D_1 \cup D_\Gamma$, $D'_2 = D_2 \cup \bar{D}_\Gamma$ and $D'_3 = D_3 \cup \hat{D}_\Gamma$.

Then we show that the push-forward of $c_{top}(\pi_{X_a}^*\mathbf{W}_{canon} \otimes \mathcal{O}(-D'_a))$ to $\mathcal{A}.(X)$ have identical components in $\mathcal{A}.(X \times_{M_n} Y(\Gamma))$ for $a = 1, 2, 3$.

Following the previous convention let $i_{h_a(D'_a)} : h_a(D'_a) \hookrightarrow X$ be the inclusions into X.

Lemma 24 *Let $D'_1 = D_1 \cup D_\Gamma$, $D'_2 = D_2 \cup \bar{D}_\Gamma$ and $D'_3 = D_3 \cup \hat{D}_\Gamma$ be defined above. Then for $a = 1, 2, 3$, the components in $X \times_{M_n} Y(\Gamma)$ of the push-forwarded Segre classes $i_{h'_a(D'_a)*}s(h'_a(D'_a), X)$ are all equal.*

Proof: From the argument in Case I, we know that we only need to prove that $h'_a(D'_a) \times_{M_n} Y(\Gamma) = h_a(D'_a) \cap \big(X \times_{M_n} Y(\Gamma)\big)$ are all equal. We notice that for all three $'a'$ we have $h'_1(D'_1) = h'_1(D_1) \cup h'_1(D_\Gamma)$, $h'_2(D'_2) = h'_2(D_2) \cup h'_2(\bar{D}_\Gamma)$, and $h'_3(D'_3) = h'_3(D_3) \cup h_3(\hat{D}_\Gamma)$.

Firstly we notice that $h'_a(D_a)$ are nothing but the $h_a(D_a)$ in Case I. On the other hand, despite that D_Γ, $\bar{D}_\Gamma$, $\hat{D}_\Gamma$ are different exceptional divisors blown up from the strict transformations of $Z(s_{canon}) \times_{M_n} Y(\Gamma)$ in three mutually birational spaces $X_\Gamma, \bar{X}_\Gamma, \hat{X}_\Gamma$, their images under h'_1, h'_2 and h'_3 are identical and their common image is $Z(s_{canon}) \times_{M_n} Y(\Gamma) = \mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y(\Gamma)$. So by combining these conclusions we have $h'_a(D'_a) = h_a(D_a) \cup \big(Z(s_{canon}) \times_{M_n} Y(\Gamma)\big)$.

Thus $h'_a(D'_a) \times_{M_n} Y(\Gamma)$ are nothing but $h_a(D_a) \times_{M_n} Y(\Gamma) \cup \big(Z(s_{canon}) \times_{M_n} Y(\Gamma)\big)$. By lemma 23 the sub-schemes $h_a(D_a) \times_{M_n} Y(\Gamma)$ has been shown to be $a-$independent, so we conclude that $h'_a(D'_a) \times_{M_n} Y(\Gamma)$ are $a-$independent as well. $\Box$

[75] This is consistent with our earlier convention.

Once we identify the $X \times_{M_n} Y(\Gamma)$ components of their Segre classes, the rest of the proof is almost identical to Case I. We omit the details. The proof of proposition 16 is finished. $\Box$

6.2 The Identification of the Localized Contribution with the Modified Algebraic Family Invariant

In this subsection, we proceed to identify the integral of the intersection pairing of the localized contribution of top Chern class with the modified algebraic family Seiberg-Witten invariant defined in section 5.2.

The push-forward into $\mathcal{A}.(X)$ of the localized contribution of the top Chern class defines a cycle class of grade $dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{W}_{canon}$. In order to get a numerical invariant $\in \mathbf{Z}$, we can either pair it with the suitable power of the tautological class $c_1(\mathbf{H})^{dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{W}_{canon}}$ on the projective space bundle $X = \mathbf{P}(\mathbf{V}_{canon})$ and push-forward the resulting class into $\mathcal{A}_0(pt) \cong \mathbf{Z}$, or we may fix a point $t_L \in T(M)$ and pair the push-forward of the localized contribution of top Chern class with $c_1(\mathbf{H})^{dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{W}_{canon} - q} \cap [t_L]$ and then [76] push it forward into $\mathcal{A}_0(pt)$. Over here $[t_L]$ represents the zero dimensional cycle class of the point t_L and the integer $q = dim_{\mathbf{C}}T(M)$ denotes the irregularity of the algebraic surface.

Now we are ready to identify the yet-to-be-enumerated intersection pairing involving the localized contribution of the top Chern class,

$$\sum_{1 \leq i \leq rank_{\mathbf{C}}\mathbf{W}_{canon}} (-1)^{i-1} c_{rank_{\mathbf{C}}\mathbf{W}_{canon}-i}(\pi_{\tilde{X}}^* \mathbf{W}_{canon} \otimes \mathbf{H} \otimes_{\Gamma' \in I_\Gamma} \mathcal{O}(-D_{\Gamma'})|_{D_\Gamma}) D_\Gamma^{i-1}[D_\Gamma] \cap c_1(\mathbf{H})^{dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{W}_{canon}}$$

with the modified mixed algebraic family Seiberg-Witten invariant

$$\mathcal{AFSW}^*_{M_{n+1} \times_{M_n} Y(\Gamma) \times T(M) \mapsto Y(\Gamma) \times T(M)}(c_{total}(\tau_\Gamma), C - \mathbf{M}(E)E - \sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i).$$

And identify

$$\sum_{1 \leq i \leq rank_{\mathbf{C}}\mathbf{W}_{canon}} (-1)^{i-1} c_{rank_{\mathbf{C}}\mathbf{W}_{canon}-i}(\pi_{\tilde{X}}^* \mathbf{W}_{canon} \otimes \mathbf{H} \otimes_{\Gamma' \in I_\Gamma} \mathcal{O}(-D_{\Gamma'})|_{D_\Gamma}) D_\Gamma^{i-1}[D_\Gamma] \cap c_1(\mathbf{H})^{dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{W}_{canon} - q} \cap [t_L]$$

with the $T(M)$-restricted version of modified mixed algebraic family Seiberg-Witten invariant $\mathcal{AFSW}^*_{M_{n+1} \times_{M_n} Y(\Gamma) \times \{t_L\} \mapsto Y(\Gamma) \times \{t_L\}}(c_{total}(\tau_\Gamma), C - \mathbf{M}(E)E - \sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i)$.

Because the identification of the latter objects is completely identical to the identification of the former, if we replace $c_1(\mathbf{H})^{dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{W}_{canon} - q} \cap [t_L]$ by $c_1(\mathbf{H})^{dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{W}_{canon}}$, we will discuss only the former case in the proof.

[76] It depends on whether we counts curves in the non-linear or linear systems.

Please consult subsection 5.2 for the definitions of the modified algebraic family Seiberg-Witten invariants and the construction of τ_Γ.

The main tools we will adopt are the machineries developed in subsection 3.2 (proposition 5) and section 4 (proposition 8 , 9, 10 and lemma 11).

Proposition 18 *Given an n-vertex admissible graph $\Gamma \in \Delta(n) \subset adm(n)$, the integration into $\mathcal{A}_0(pt) \cong \mathbf{Z}$ of $c_1(\mathbf{H})^{dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{W}_{canon}}$ capping with the push-forward of the localized contribution of the top Chern class along the blown up divisor D_Γ,*

$$\sum_{1\leq i\leq rank_{\mathbf{C}}\mathbf{W}_{canon}} (-1)^{i-1} c_{rank_{\mathbf{C}}\mathbf{W}_{canon}-i}(\pi_{\tilde{X}}^{*}\mathbf{W}_{canon}\otimes\mathbf{H}\otimes_{\Gamma'\in I_\Gamma}\mathcal{O}(-D_{\Gamma'})|_{D_\Gamma})D_\Gamma^{i-1}[D_\Gamma]$$

in $\mathcal{A}.(X)$, is equal to the modified mixed algebraic family invariant $\mathcal{AFSW}^{*}_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_t$ $\mathbf{M}(E)E - \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$ defined in subsection 5.2.

Likewise for an arbitrary point $t_L \in T(M)$, the integral of the pairing of $c_1(\mathbf{H})^{dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{W}_{canon}-q} \cap [t_L]$ with the image of the above localized contribution of top Chern class along D_Γ into $\mathcal{A}.(X)$, is equal to the modified mixed algebraic family invariant

$$\mathcal{AFSW}^{*}_{M_{n+1}\times_{M_n}Y(\Gamma)\times\{t_L\}\mapsto Y(\Gamma)\times\{t_L\}}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i).$$

Because the t_L-restricted version is completely parallel to the non-restricted version, we only offer a proof for the non-restricted version. Please consult remark 19 on page 95 right after the end of the proof.

Proof of proposition 18: The proof of the proposition involves an induction on the element $\Gamma \in \Delta(n)$ based on the linear ordering $\models$ (see page 53 for the recursive definition of $\models$).

Firstly, we provide a simple computation on the dimension formula which motivates the appearance of τ_Γ in the modified family invariant. Because $X = \mathbf{P}(\mathbf{V}_{canon}) \mapsto M_n \times T(M)$, $dim_{\mathbf{C}}X = rank_{\mathbf{C}}\mathbf{V}_{canon} - 1 + dim_{\mathbf{C}}M_n + q$. Thus, $dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{W}_{canon} = dim_{\mathbf{C}}M_n + q + rank_{\mathbf{C}}(\mathbf{V}_{canon} - \mathbf{W}_{canon}) - 1$. Based on the fact that $(\Phi_{\mathbf{V}_{canon}\mathbf{W}_{canon}}, \mathbf{V}_{canon}, \mathbf{W}_{canon})$ is the canonical algebraic family Kuranishi model of the class $C - \mathbf{M}(E)E$, we know that

$$rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon}) = 1-q+p_g+\frac{(C-\mathbf{M}(E)E)^2 - c_1(\mathbf{K}_{M_{n+1}/M_n})\cdot(C-\mathbf{M}(E)E)}{2}$$

$$= 1 - q + p_g + \frac{C^2 - c_1(\mathbf{K}_M)\cdot C - \sum_{1\leq i\leq n}(m_i^2 + m_i)}{2},$$

by surface Riemann-Roch formula. From this we can infer the relationship between the raised power of $c_1(\mathbf{H})$ in the intersection pairing (which is also the

expected algebraic family dimension of $\mathcal{M}_{C-\mathbf{M}(E)E}$) and the singular multiplicities $m_i, 1 \leq i \leq n$.

On the other hand, by a direct computation the expected (family algebraic) dimension of the space $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i} \times_{M_n} Y(\Gamma)$ is given by

$$p_g + dim_{\mathbf{C}} Y(\Gamma) + \frac{(C-\mathbf{M}(E)E)^2 - c_1(\mathbf{K}_{M_{n+1}/M_n})\cdot(C-\mathbf{M}(E)E)}{2} - \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} (C-\mathbf{M}(E)E)\cdot e_i$$

$$+\frac{(\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)^2 + c_1(\mathbf{K}_{M_{n+1}/M_n})\cdot(\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)}{2}.$$

By using $dim_{\mathbf{C}} Y(\Gamma) = dim_{\mathbf{C}} M_n + \sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} \frac{e_i^2 - c_1(\mathbf{K}_{M_{n+1}/M_n})\cdot e_i}{2}$, the above expression can be simplified to

$$= p_g + dim_{\mathbf{C}} M_n + \frac{(C-\mathbf{M}(E)E)^2 - c_1(\mathbf{K}_{M_{n+1}/M_n})\cdot(C-\mathbf{M}(E)E)}{2}$$

$$+\{\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i\cdot\big(e_i + \sum_{j<i; e_j\cdot(C-\mathbf{M}(E)E)<0} e_j - (C-\mathbf{M}(E)E)\big)\}.$$

A direct comparison with the formula of $rank_{\mathbf{C}}\tau_\Gamma$ shows that this correction term matches up with the rank of $\tau_\Gamma \otimes \mathbf{H}$ found in subsection 5.2 lemma 17. This explains morally why we need to insert $c_{top}(\tau_\Gamma \otimes \mathbf{H}) = \sum_{l\leq rank_{\mathbf{C}}\tau_\Gamma} c_l(\tau_\Gamma) \cap c_1(\mathbf{H})^{rank_{\mathbf{C}}\tau_\Gamma - l}$ in the corresponding modified algebraic family invariant. The dimension count singles out the role of τ_Γ-as a mean to **compensate the discrepancy of the expected family dimensions** between $\mathcal{M}_{C-\mathbf{M}(E)E}$ and $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i} \times_{M_n} Y(\Gamma)$. In the latter half of the proof, we will see why a correct choice of τ_Γ (not only the rank itself) is essential in our identification.

We start from the simplest case when $\Gamma \in \Delta(n)$ is a minimal element of $\Delta(n)$ under $\models$. Under this assumption, $I_\Gamma = \emptyset$ and Γ is a minimal element under $\succ$. The minimality assumption of Γ under $\succ$ implies that there can be no Γ' with $S_{\Gamma'} \cap Y(\Gamma) \neq \emptyset$. In such a case the space $S_\Gamma \subset Y(\Gamma)$ (over which the type I exceptional cone $\mathcal{C}_\Gamma$ is constant) itself is a closed subset of M_n, and therefore is equal to $Y(\Gamma)$. The consequence $S_\Gamma = Y(\Gamma)$ implies that all the type I exceptional curves dual to e_i, with $e_i\cdot(C-\mathbf{M}(E)E) < 0$, remain smooth and irreducible throughout the whole $Y(\Gamma)$. In particular, no curves dual to such e_i can break into more than one irreducible component over $Y(\Gamma)$.

By lemma 11, the sum [77] $\sum_{1\leq i\leq rank_{\mathbf{C}}\mathbf{W}_{canon}} (-1)^{i-1} c_{rank_{\mathbf{C}}\mathbf{W}_{canon}-i}(\pi_{\tilde{X}}^* \mathbf{W}_{canon} \otimes \mathbf{H}) D_\Gamma^{i-1}[D_\Gamma]$ is nothing but the localized contribution of the top Chern class defined in section 6 of [Liu5]. By proposition 11 of [Liu5] and our knowledge that

[77] It is simplified as there is no blowing up ahead of the one parametrized by Γ.

the type I exceptional curves dual to e_i, with $e_i \cdot (C - \mathbf{M}(E)E) < 0$ remain irreducible and smooth throughout $Y(\Gamma)$, these imply that the natural bundle map $\pi_X^*\mathbf{W}^\circ_{canon} \otimes \mathbf{H}|_{X\times_{M_n}Y(\Gamma)\times T(M)} \mapsto \pi_X^*\mathbf{W}_{canon}\otimes \mathbf{H}|_{X\times_{M_n}Y(\Gamma)\times T(M)}$ (see proposition 9 of [Liu5] for its construction) is injective over the whole $X\times_{M_n} Y(\Gamma)\times T(M)$. In terms of the notation of proposition 7 of the current paper or proposition 12/corollary 3 of [Liu5], the union of cones $\cup_{i>0}\mathbf{C}_{\rho_i}$ corresponding the kernel of the bundle map is empty. Thus the **simplifying assumption** in section 6.1 of [Liu5] has been satisfied automatically because the restricted family moduli space $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i}\times_{M_n} Y(\Gamma) = Z(s^\circ_{canon})\times_{M_n} Y(\Gamma)$ does not intersect with $\cup_{i>0}\mathbf{C}_{\rho_i} = \emptyset$ at all. The argument of theorem 4 of [Liu5] is then applicable and we may identify the integration of the top intersection pairing of the localized top Chern class along D_Γ (over $Y(\Gamma)$) and $c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}$ to [78] be $\mathcal{AFSW}_{M_{n+1}\times T(M)\times_{M_n}Y(\Gamma)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$, which is nothing but the modified invariant $\mathcal{AFSW}^*_{M_{n+1}\times T(M)\times_{M_n}Y(\Gamma)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0} e_i)$ by definition 12. Consult section 6.1-6.4 of [Liu5] for the details of the identification. [79]

Next we consider the general (and a priori more complicated) situation when Γ is not minimal under $\models$.

Induction Hypothesis: Assuming that for all the $\Gamma' \in I_\Gamma$ (i.e. $\Gamma \models \Gamma'$), the integral of the following top intersection pairing with localized contribution of top Chern class

$$\sum_{1\leq i\leq rank_{\mathbf{C}}\mathbf{W}_{canon}} (-1)^{i-1} c_{rank_{\mathbf{C}}\mathbf{W}_{canon}-i}(\pi^*_{\tilde{X}}\mathbf{W}_{canon}\otimes\mathbf{H}\otimes_{\Gamma''\in I_{\Gamma'}}\mathcal{O}(-D_{\Gamma''})|_{D_{\Gamma'}})$$

$$\cap D_{\Gamma'}^{i-1}[D_{\Gamma'}]\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}$$

have been identified with the modified algebraic family invariant,

$$\mathcal{AFSW}^*_{M_{n+1}\times_{M_n}Y(\Gamma')\times T(M)\mapsto Y(\Gamma')\times T(M)}(c_{total}(\tau_{\Gamma'}), C-\mathbf{M}(E)E-\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0} e'_i).$$

As usual e'_i are the type I exceptional classes over $Y_{\Gamma'}$ and $\tau_{\Gamma'}$ is the associated tau class defined for Γ' by definition 10 on page 56.

By proposition 16, one may "collapse" the blowing up sequence indexed by I_Γ (following the reversed ordering of $\models$) to the new blowing up sequence

[78] We skip the push-forward operation into $\mathcal{A}_\cdot(X)$ on the localized top Chern class by interpreting the cap product with the complementary power of $c_1(\mathbf{H})$ as capping this natural Chern class pull-back by $D_\Gamma \mapsto X$. To simplify our notations, we will adopt the same convention afterward. The reader should be able to recover it from the context.

[79] In the following inductive argument, a specialization of our argument for the general case also provides a proof for the special case.

indexed by the reduced index set $\bar{I}_\Gamma$ (see definition 16 for its definition) following the reversed linear ordering of $\vdash$ without changing the answer. As was argued, the permutation and the collapsing of blowing up centers not affect the result of the localized contribution of top Chern class. Thus the yet-to-be-identified intersection pairing is equal to [80]

$$\{\sum_{i=1}^{i=rank_{\mathbf{C}}\mathbf{W}_{canon}}(-1)^{i-1}c_{rank_{\mathbf{C}}\mathbf{W}_{canon}-i}(\pi^*_{\tilde{\hat{X}}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}\otimes_{\Gamma'\in\bar{I}_\Gamma}\mathcal{O}(-\hat{D}_{\Gamma'}))\cap\hat{D}_\Gamma^{i-1}\cap[\hat{D}_\Gamma]\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}\}.$$

Among the many different blowing ups indexed by the graphs $\Gamma'\in\bar{I}_\Gamma$, whenever $\Gamma'\in\bar{I}_\Gamma^{\gg}$ the restricted family moduli spaces $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i}\times_{M_n} Y(\Gamma')$ can be viewed naturally as sub-schemes of $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i}\times_{M_n} Y(\Gamma)$ (by adjoining curves in $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i}$ to the exceptional curves dual to $\sum_{e_i\cdot(C-\mathbf{M}(E)E)>0;e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i$) and therefore sub-scheme of $\mathcal{M}_{C-\mathbf{M}(E)E}\times_{M_n} Y(\Gamma)$.

Under the reversed $\vdash$ linear ordering of blowing ups, these blowing ups with $\Gamma'\in\bar{I}_\Gamma^{\gg}$ are performed at the very end of the linear chain of blowing ups parametrized by $\bar{I}_\Gamma$. Thus we may decompose $\bar{I}_\Gamma=\bar{I}_\Gamma^{\gg}\coprod(\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg})$ and use residual intersection formula to re-write the above intersection pairing as

$$\{\sum_{i=1}^{i=rank_{\mathbf{C}}\mathbf{W}_{canon}}(-1)^{i-1}c_{rank_{\mathbf{C}}\mathbf{W}_{canon}-i}(\pi^*_{\tilde{\hat{X}}_\Gamma}\mathbf{W}_{canon}\otimes\mathbf{H}\otimes_{\Gamma'\in\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg}}\mathcal{O}(-\hat{D}_{\Gamma'}))$$

$$\cap\hat{D}_\Gamma^{i-1}\cap[\hat{D}_\Gamma]\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}\}$$

$$-\sum_{\Gamma'\in\bar{I}_\Gamma^{\gg}}\{\sum_{i=1}^{i=rank_{\mathbf{C}}\mathbf{W}_{canon}}(-1)^{i-1}c_{rank_{\mathbf{C}}\mathbf{W}_{canon}-i}(\pi^*_{\tilde{\hat{X}}_{\Gamma'}}\mathbf{W}_{canon}\otimes\mathbf{H}\otimes_{\Gamma_1\in\bar{I}_\Gamma;\Gamma'\vdash\Gamma_1}\mathcal{O}(-\hat{D}_{\Gamma_1}))$$

$$\cap\hat{D}_{\Gamma'}^{i-1}\cap[\hat{D}_{\Gamma'}]\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}\}.$$

In the second term of the above sum (i.e. when $\Gamma'\in\bar{I}_\Gamma^{\gg}$), the index set restriction "$\Gamma_1\in\bar{I}_\Gamma;\Gamma'\vdash\Gamma_1$" is the same as the alternative restriction "$\Gamma_1\in(\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg})\coprod\{\Gamma_1|\Gamma_1\in\bar{I}_\Gamma^{\gg};\Gamma'\gg\Gamma_1\}$". [81]

By the definition/construction of $\bar{I}_\Gamma$ and $\bar{I}_\Gamma^{\gg}$, all elements $\Gamma''\in(\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg})$ satisfy $(\Gamma,\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i)\sqsupset(\Gamma'',\sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0}e''_i)$. Thus $(\Gamma',\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i)\sqsupset$

[80] The space $\tilde{\hat{X}}_\Gamma$ with the exceptional divisor $\hat{D}_\Gamma$ denotes the blowing up of $\hat{X}_\Gamma$. The hatted divisor $\hat{D}_{\Gamma'}$ have been used in the previous subsection already to denote the exceptional divisors blown up following the reversed linear ordering of $\vdash$ inside $\bar{I}_\Gamma$.

[81] It is because $\vdash$ is identical to $\gg$ on the subset $\bar{I}_\Gamma^{\gg}$ and by the definition of $\vdash$ (see definition 17) the elements in $\bar{I}_\Gamma^{\gg}$ are larger than all elements in $\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg}$)

$(\Gamma'', \sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0} e''_i)$ as well since we know $\Gamma'\in \bar{I}_{\Gamma}^{\gg}$. By the discussion of lemma 22, the alternative index set restriction on Γ_1 can then be replaced by the equivalent one "$\Gamma_1\in \bar{I}_{\Gamma'}$". Then by applying proposition 16 to $I_{\Gamma'}$ and $\bar{I}_{\Gamma'}$, we may "un-collapse" to restore the reduced index set $\bar{I}_{\Gamma'}$ back to $I_{\Gamma'}$ without affecting the result of the intersection pairing. This implies that the second term of the above expression can be expressed as (observe that the hat of $\hat{D}_{\Gamma'}$ or $\hat{D}_{\Gamma_1}$ has been removed)

$$-\sum_{\Gamma'\in \bar{I}_{\Gamma}^{\gg}}\{\sum_{i=1}^{i=rank_{\mathbf{C}}\mathbf{W}_{canon}}(-1)^{i-1}c_{rank_{\mathbf{C}}\mathbf{W}_{canon}-i}(\pi_{\tilde{X}}^{*}\mathbf{W}_{canon}\otimes\mathbf{H}\otimes_{\Gamma_1\in I_{\Gamma'}}\mathcal{O}(-D_{\Gamma_1}))$$

$$\cap D_{\Gamma'}^{i-1}\cap[D_{\Gamma'}]\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}\}.$$

Then by the **Inductive Hypothesis** on page 83 above, the integral of each of these terms is equal to a modified algebraic family invariant attached to Γ' and the original yet-to-be-identified intersection pairing of the localized contribution of the top Chern class with $c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}$ is equal to [82]

$$\int_{\tilde{\tilde{X}}_{\Gamma}}\{\sum_{i=1}^{i=rank_{\mathbf{C}}\mathbf{W}_{canon}}(-1)^{i-1}c_{rank_{\mathbf{C}}\mathbf{W}_{canon}-i}(\pi_{\tilde{\tilde{X}}_{\Gamma}}^{*}\mathbf{W}_{canon}\otimes\mathbf{H}\otimes_{\Gamma'\in\bar{I}_{\Gamma}-\bar{I}_{\Gamma}^{\gg}}\mathcal{O}(-\hat{D}_{\Gamma'}))$$

$$\cap\hat{D}_{\Gamma}^{i-1}\cap[\hat{D}_{\Gamma}]\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}\}$$

$$-\sum_{\Gamma'\in\bar{I}_{\Gamma}^{\gg}}\mathcal{AFSW}^{*}_{M_{n+1}\times_{M_n}Y(\Gamma')\times T(M)\mapsto Y(\Gamma')\times T(M)}(c_{total}(\tau_{\Gamma'}),C-\mathbf{M}(E)E-\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i).$$

If we can identify the first term in this sum with

$$\mathcal{AFSW}_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_{\Gamma}),C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i),$$

then by definition 13, the total expression is exactly what was defined to be the modified algebraic family invariant attached to Γ,

$$\mathcal{AFSW}^{*}_{M_{n+1}\times_{M_n}Y(\Gamma)\times T(M)\mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_{\Gamma}),C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i)$$

and then the identification is complete.

[82]This inductive pattern is exactly why we had defined the modified invariants earlier on page 58 in this way.

In the rest of the proof we identify the first term of the last sum with the specific mixed algebraic family Seiberg-Witten invariant associated with $c_{total}(\tau_\Gamma)$.

Following the same convention as in [Liu5], we let $k_1 < k_2 < \cdots < k_p$ be the subscripts such that $e_{k_i} \cdot (C - \mathbf{M}(E)E) < 0$ for all $1 \le i \le p$.

Case I: In this case we deal with the more interesting situation that $e_{k_i}^2 \ge e_{k_i} \cdot (C - \mathbf{M}(E)E)$ for all $1 \le i \le p$.

Step One: Firstly we make use of the assumption $e_{k_i}^2 \ge e_{k_i} \cdot (C - \mathbf{M}(E)E)$, $1 \le i \le p$ and show the following Chern classes identity for $\mathbf{V}_{quot}$,

Lemma 25 *Let $\pi_g : X \times_{M_n} Y(\Gamma) \mapsto Y(\Gamma) \times T(M)$ and $\pi_t : Y(\Gamma) \times T(M) \mapsto Y(\Gamma)$ be the natural projection maps. Let $\mathbf{V}_{quot}$ be the quotient bundle of $\mathbf{W}_{canon}|_{Y(\Gamma)\times T(M)}$ as was defined on page 21. Let τ_Γ be the tau class defined in definition 10 on page 56.*

Suppose that $e_{k_i}^2 \ge e_{k_i} \cdot (C - \mathbf{M}(E)E)$ for all $1 \le i \le p$, then there is an identity among the top Chern classes over $X \times_{M_n} Y(\Gamma)$,

$$c_{top}(\mathbf{H} \otimes \pi_g^* \mathbf{V}_{quot}) = c_{top}(\mathbf{H} \otimes \pi_g^* \tau_\Gamma) \cap c_{top}(\pi_g^* \pi_t^* \mathbf{N}_{Y(\Gamma)} X).$$

Proof: Let $\mathbf{Q}_{k_i}$ and $\mathbf{E}_C$ be the line bundles associated to the invertible sheaves $\mathcal{Q}_{k_i}$ and $\mathcal{E}_C$ which appeared in definition 10 (see also proposition 12).

Define the vector bundle $\mathbf{G}_\Gamma = \mathbf{H} \otimes \pi_g^* \oplus_{1 \le i \le p} \pi_t^* \mathbf{N}_{Y(\Gamma_{e_{k_i}})} M_n|_{Y(\Gamma)} \otimes \mathbf{Q}_{k_i} \otimes \mathbf{E}_C \mapsto X \times_{M_n} Y(\Gamma)$ to be a rank $dim_{\mathbf{C}} M_n - dim_{\mathbf{C}} Y(\Gamma) = codim_{\mathbf{C}} \Gamma$ vector bundle over $X \times_{M_n} Y(\Gamma)$ [83].

By prop 5 on page 31 in subsection 3.2, definition 4 on page 36 and the tensor product formula of the top Chern classes, we know $c_{top}(\mathbf{H} \otimes \pi_g^* \mathbf{V}_{quot}) = c_{top}(\mathbf{H} \otimes \pi_g^* \tilde{\mathbf{V}}_{quot})$.

Thus we have

$$c_{top}(\mathbf{H} \otimes \pi_g^* \mathbf{V}_{quot}) = c_{top}((\mathbf{H} \otimes \pi_g^* \tilde{\mathbf{V}}_{quot} - \mathbf{G}_\Gamma) \oplus \mathbf{G}_\Gamma).$$

Recall that by proposition 4 in section [84] 2 the space $Y(\Gamma) = \cap_{1 \le i \le p} Y(\Gamma_{e_{k_i}})$ is a transversal intersection of smooth loci $Y(\Gamma_{e_i})$. We also know from lemma 9 of [Liu5] that all the $Y(\Gamma_{e_{k_i}})$, being the family moduli space of e_{k_i}, are defined by regular global sections of $\mathcal{R}^0 \pi_* \big(\mathcal{O}_{\sum_{j_{k_i}} E_{j_{k_i}}}(E_{k_i})\big)$ over M_n determined by the morphism of locally free sheaves [85]

$$\mathcal{R}^0 \pi_* \mathcal{O}_{E_{k_i}} \mapsto \mathcal{R}^0 \pi_* \big(\mathcal{O}_{\sum_{j_{k_i}} E_{j_{k_i}}}(E_{k_i})\big)$$

[83] Notice that $\mathbf{G}_\Gamma$ is constructed from $\mathbf{N}_{X \times_{M_n} Y(\Gamma)} X$ twisted by $\mathbf{Q}_{k_i}$ and $\mathbf{E}_C$ on direct direct factors.

[84] Or equivalently by proposition 4.7 on page 426 of [Liu1].

[85] The former is invertible, while the latter is the canonical obstruction bundle of e_{k_i}.

and $\mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum_{j_{k_i}} E_{j_{k_i}}}(E_{k_i})|_{Y(\Gamma_{e_{k_i}})}\big) \cong \mathcal{R}^1\pi_*\big(\mathcal{O}_{\tilde{\Xi}_{k_i}}(E_{k_i}-\sum_{j_{k_i}} E_{j_{k_i}})\big) \cong \mathcal{N}_{Y(\Gamma_{e_{k_i}})}M_n$, the normal sheaf of $Y(\Gamma_{e_{k_i}})$ in M_n.

By definition 10, lemma 17 and definition 4, the class $\tilde{\mathbf{V}}_{quot} - \oplus_{1\leq i\leq p}\mathbf{N}_{Y(\Gamma_{e_{k_i}})}M_n|_{Y(\Gamma)} \otimes \mathbf{Q}_{k_i} \otimes \mathbf{E}_C$ is equal to τ_Γ (expressed here by a difference of vector bundles rather than the corresponding locally free sheaves), and is represented by a vector bundle of rank $rank_{\mathbf{C}}\tilde{\mathbf{V}}_{quot} - codim_{\mathbf{C}}Y(\Gamma)$. This and the Whitney sum formula of the top Chern class imply that $c_{top}(\mathbf{H}\otimes\pi_g^*\mathbf{V}_{quot}) = c_{top}(\mathbf{H}\otimes\pi_g^*\tau_\Gamma)\cap c_{top}(\mathbf{G}_\Gamma)$.

Because for the type I exceptional class $e_{k_i} = E_{k_i} - \sum_{j_{k_i}} E_{j_{k_i}}$ the smooth locus $Y(\Gamma_{e_{k_i}}) \subset M_n$ has been the regular zero locus defined by the canonical global section of $\mathcal{W}_{e_{k_i}} = \mathcal{R}^0\pi_*\big(\mathcal{O}_{\sum_{j_{k_i}} E_{j_{k_i}}}(E_{k_i})\big)$, defined by the canonical algebraic Kuranishi model of e_{k_i} (see section 6.2 of [Liu5]). So $\mathcal{W}_{e_{k_i}}|_{Y(\Gamma_{e_{k_i}})} \cong \mathcal{N}_{Y(\Gamma_{e_{k_i}})}M_n$. By lemma 10 of [86] [Liu5], this implies that the top Chern class $c_{top}(\pi_g^*(\pi_t^*\mathbf{N}_{Y(\Gamma_{e_{k_i}})}M_n|_{Y(\Gamma)}\otimes\mathbf{Q}_{k_i})\otimes\mathbf{H}\otimes\mathcal{E}_C)$ on $X\times_{M_n}Y(\Gamma)$, where $Y(\Gamma) = \cap_{1\leq i\leq p}Y(\Gamma_{e_{k_i}})$, is equal to the top Chern class of the un-twisted version $c_{top}(\mathbf{N}_{\pi_g^*(\pi_t^*Y(\Gamma_{e_{k_i}})}M_n|_{Y(\Gamma)})$.

Because this is applicable to all $1\leq i\leq p$, we find that $c_{top}(\mathbf{G}_\Gamma) = c_{top}(\pi_g^*\pi_t^*\mathbf{N}_{Y(\Gamma)}M_n)$. The proof of this lemma is complete. $\Box$

Secondly we consider the normal cone (into its compactification) $\mathbf{C}_{\hat{X}_\Gamma\times_{M_n}Y(\Gamma)}\hat{X}_\Gamma \subset \mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma\times_{M_n}Y(\Gamma)}\hat{X}_\Gamma\oplus 1)$ of the closed embedding $\hat{X}_\Gamma\times_{M_n}Y(\Gamma)\subset\hat{X}_\Gamma$.

Consider [87] the blowing up along $\hat{X}_\Gamma\times_{M_n}Y(\Gamma)\times\{0\}\subset\hat{X}_\Gamma\times\mathbf{C}$. The exceptional divisor of this blowing up is isomorphic to $\mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma\times_{M_n}Y(\Gamma)}\hat{X}_\Gamma\oplus 1)$.

As it projects onto $X\times_{M_n}Y(\Gamma)\times\{0\}\subset X\times\mathbf{C}$ under $\hat{X}_\Gamma\times\mathbf{C}\mapsto X\times\mathbf{C}$, by the universal property of the blowing up (proposition 7.14. on page 164 of [Ha]), $\mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma\times_{M_n}Y(\Gamma)}\hat{X}_\Gamma\oplus 1)$ maps onto the exceptional divisor [88] $\mathbf{P}(\mathbf{N}_{X\times_{M_n}Y(\Gamma)}X\oplus 1)$ of the blowing up along $X\times_{M_n}Y(\Gamma)\times\{0\}\subset X\times\mathbf{C}$. Then it induces a surjection of the normal cones $\mathbf{C}_{\hat{X}_\Gamma\times_{M_n}Y(\Gamma)}\hat{X}_\Gamma\mapsto\mathbf{N}_{X\times_{M_n}Y(\Gamma)}X$ and we have the following commutative diagram,

$$\begin{array}{ccccc}
\hat{X}_\Gamma\times_{M_n}Y(\Gamma) & \longrightarrow & \mathbf{C}_{\hat{X}_\Gamma\times_{M_n}Y(\Gamma)}\hat{X}_\Gamma & \longrightarrow & \mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma\times_{M_n}Y(\Gamma)}\hat{X}_\Gamma\oplus 1)\\
\Big\downarrow{\scriptstyle \pi_h} & & \Big\downarrow{\scriptstyle \pi_{\mathbf{C}}} & & \Big\downarrow{\scriptstyle \pi_{\mathbf{P}}}\\
X\times_{M_n}Y(\Gamma) & \longrightarrow & \mathbf{C}_{X\times_{M_n}Y(\Gamma)}X & \longrightarrow & \mathbf{P}(\mathbf{C}_{X\times_{M_n}Y(\Gamma)}X\oplus 1)
\end{array}$$

As the exceptional divisor $\hat{D}_\Gamma$ maps into $Y(\Gamma)$ under $\tilde{\hat{X}}_\Gamma\mapsto M_n$, we have $\hat{D}_\Gamma\subset\tilde{\hat{X}}_\Gamma\times_{M_n}Y(\Gamma)$. Our original intersection pairing involving $\hat{D}_\Gamma$ can be pushed-forward by $(\pi_r)_*$ (here $\pi_r:\tilde{\hat{X}}_\Gamma\mapsto\hat{X}_\Gamma$ is the blowing down map) into

[86] This lemma implies that the restriction of the top Chern class of a vector bundle $\mathbf{E}$ to the regular zero locus $Z(s)$ with codimension $rank_{\mathbf{C}}\mathbf{E}$ of its regular section s is equal to the restriction of the top Chern class of $\mathbf{E}\otimes\mathbf{Q}$, twisted by a line bundle $\mathbf{Q}$. Notice that it holds only because we are working in $\mathcal{A}.(Z(S))$ instead of the whole space X.

[87] following chapter 5 of [F].

[88] The embedding is regular, so we use $\mathbf{N}_{X\times_{M_n}Y(\Gamma)}X$ and $\mathbf{C}_{X\times_{M_n}Y(\Gamma)}X$ interchangeably.

$\mathcal{A}_0(\hat{X}_\Gamma \times_{M_n} Y(\Gamma))$ and it defines a zero dimensional cycle class lying inside $\hat{X}_\Gamma \times_{M_n} Y(\Gamma)$.

Either by a direct computation on the localized contribution of top Chern class (involving the Segre class of some normal cone), or by the technique of the deformation to the normal cone (consult chapter 5 of [F]) from $\hat{X}_\Gamma \times_{M_n} Y(\Gamma) \subset \hat{X}_\Gamma$ to $\hat{X}_\Gamma \times_{M_n} Y(\Gamma) \subset \mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma$, one may replace the total space $\hat{X}_\Gamma$ by a linearized object of the same dimension, namely the projectified normal cone $\mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma \oplus 1)$ or its affine part [89] $\mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma$.

Define $\pi_f : \mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma \oplus 1) \mapsto \hat{X}_\Gamma \times_{M_n} Y(\Gamma)$ to be the projection map. To get a global intersection pairing on $\mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma \oplus 1)$ which is refined to our localized intersection pairing, we twist the bundle $\pi_f^*\pi_g^*\mathbf{W}_{canon}$ by $\mathcal{O}(\hat{\mathbf{P}}_\infty)$, where $\hat{\mathbf{P}}_\infty = \mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma) \subset \mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma \oplus 1)$ is the divisor at infinity. As our intersection pairing is localized at the zero section $\hat{X}_\Gamma \times_{M_n} Y(\Gamma)$ (totally disjoint from $\hat{\mathbf{P}}_\infty$), the fact that $\mathcal{O}(\hat{\mathbf{P}}_\infty)$ is trivialized over the affine cone $\mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma$ allows us to remove the $\mathcal{O}(\hat{\mathbf{P}}_\infty)$ tensor product effectively in our calculation [90].

Consider the pull-back of the $\mathbf{H}$-twisted short exact sequence $0 \mapsto \underline{\mathbf{W}}_{canon}| \mapsto \mathbf{W}_{canon}|_{Y(\Gamma)\times T(M)} \mapsto \mathbf{V}_{quot} \mapsto 0$ (consult page 21 in section 3) by $(\pi_g\pi_h\pi_f)^*$, [91] the short exact sequence exists on the whole space $\mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma \oplus 1)$. [92]

By proposition 10 we know that the residual intersection formula of $\mathbf{W}_{canon}$ and of $\underline{\mathbf{W}}_{canon}$ are compatible. We may replace the above top intersection pairing of the localized Chern class by

$$\{c_{total}((\pi_h\pi_f)^*(\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathbf{H})\otimes_{\Gamma'\in \bar{I}_\Gamma - \bar{I}_\Gamma^{\gg}}\mathcal{O}(-\pi_f^*\hat{D}_{\Gamma'}))\cap s_{total}(\pi_r(\hat{D}_\Gamma), \mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma)\}_{dim_{\mathbf{C}}X - rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}}$$

$$\cap c_{top}((\pi_h\pi_f)^*(\mathbf{H}\otimes\pi_g^*\mathbf{V}_{quot}))\cap c_1((\pi_h\pi_f)^*\mathbf{H})^{dim_{\mathbf{C}}M_n + rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}.$$

By lemma 25 we may replace the top Chern class $c_{top}((\pi_h\pi_f)^*(\mathbf{H}\otimes\pi_g^*\mathbf{V}_{quot}))$ by $c_{top}((\pi_h\pi_f)^*(\mathbf{H}\otimes\pi_g^*\tau_\Gamma))\cap c_{top}((\pi_t\pi_g\pi_h\pi_f)^*\mathbf{N}_{Y(\Gamma)}M_n)$, which is the same as $c_{top}((\pi_h\pi_f)^*(\mathbf{H}\otimes\pi_g^*\tau_\Gamma))\cap c_{top}((\pi_h\pi_f)^*\mathbf{N}_{X\times_{M_n} Y(\Gamma)}X)$.

On the other hand the projection of normal cones $\pi_{\mathbf{C}} : \mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma \mapsto \mathbf{N}_{X\times_{M_n} Y(\Gamma)}X$ induces (by pull-back) a tautological section of $(\pi_h\pi_f)^*\mathbf{N}_{X\times_{M_n} Y(\Gamma)}X$ over $\mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma$ and [93] its zero locus is exactly $\hat{X}_\Gamma \times_{M_n} Y(\Gamma) \subset \mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma$.

Therefore the cycle class $[\hat{X}_\Gamma \times_{M_n} Y(\Gamma)] \subset \mathcal{A}.(\mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma)$ represents the cap product of the fundamental class with the top Chern class $c_{top}((\pi_h\pi_f)^*\mathbf{N}_{X\times_{M_n} Y(\Gamma)}X)$.

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[89]We prefer the former if we want the space to be complete.

[90]as far as our intersection cycle does not overlap with $\hat{\mathbf{P}}_\infty$.

[91]The map π_h was defined in the above commutative diagram on page 87.

[92]This is the benefit of adopting the projectified normal cone $\mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma \oplus 1)$ than the original space $\hat{X}_\Gamma$, a replacement of tubular neighborhood in the $\mathcal{C}^\infty$ category.

[93]It extends to a section of $(\pi_h\pi_f)^*\mathbf{N}_{X\times_{M_n} Y(\Gamma)}X \otimes \mathcal{O}(\hat{\mathbf{P}}_\infty)$ on $\mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma)}\hat{X}_\Gamma \oplus 1)$.

Thus we may replace this top Chern class $c_{top}((\pi_h\pi_f)^*\mathbf{N}_{X\times_{M_n}Y(\Gamma)}X)$ in our pairing by the zero section cycle class $[\hat{X}_\Gamma\times_{M_n}Y(\Gamma)]$ of the compactification of the normal cone. Consequently, we can restrict the intersection pairing to the zero section $\hat{X}_\Gamma\times_{M_n}Y(\Gamma)$ of its normal cone in $\hat{X}_\Gamma$ and get

$$\{c_{total}(\pi_h^*(\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathbf{H})\otimes_{\Gamma'\in\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg}}\mathcal{O}(-\hat{D}_{\Gamma'}))\cap s_{total}(\pi_r(\hat{D}_\Gamma),\mathbf{C}_{\hat{X}_\Gamma\times_{M_n}Y(\Gamma)}\hat{X}_\Gamma)\}_{dim_{\mathbf{C}}X-rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}}$$

$$\cap c_{top}(\pi_h^*(\mathbf{H}\otimes\pi_g^*\tau_\Gamma))\cap c_1(\pi_h^*\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}\in\mathcal{A}_0(\hat{X}_\Gamma\times_{M_n}Y(\Gamma)).$$

$\diamondsuit$ Let us summarize: In step one we have succeeded in restricting the top intersection pairing to the subspace $\hat{X}_\Gamma\times_{M_n}Y(\Gamma)$ by using some novel property of $\tilde{\mathbf{V}}_{quot}$. The class $c_{top}(\mathbf{H}\otimes\pi_g^*\tau_\Gamma)$ has appeared because of lemma 25.

Step Two: Consider the bundle map $\pi_h^*(\pi_g^*\mathbf{W}^\circ_{canon}|_{Y(\Gamma)\times T(M)}\otimes\mathbf{H})\mapsto\pi_h^*(\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathbf{H})$ induced by $\mathbf{H}$-twisted version of the π_g^*-pulled-back vector bundle map $\mathbf{W}^\circ_{canon}\mapsto\underline{\mathbf{W}}_{canon}$ over $Y(\Gamma)\times T(M)$ (see page 21) pull-back by π_h^*. Our goal is to explain why we may use the top Chern class of $\pi_h^*(\pi_g^*\mathbf{W}^\circ_{canon}|_{Y(\Gamma)\times T(M)}\otimes\mathbf{H})$ to replace the complicated bundle $\pi_h^*(\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathbf{H})\otimes\mathcal{O}(-\sum_{\Gamma'\in\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg}}\hat{D}_{\Gamma'})$ in the localized contribution of top Chern class.

Observe that for all $\Gamma'\in\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg}$, $(\Gamma,\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i)\sqsupset(\Gamma'',\sum_{e''_i\cdot(C-\mathbf{M}(E)E)<0}e''_i)$. This condition $\sqsupset$ (consult definition 15 for its definition) implies that the sub-scheme $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i}\times_{M_n}Y(\Gamma)\cap Y(\Gamma')$ can be embedded into $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i}\times_{M_n}Y(\Gamma)\cap Y(\Gamma')$.

Then by proposition 14 and the remark 17 right after its proof, we may decompose $\mathcal{M}_{C-\mathbf{M}(E)E}\times_{M_n}Y(\Gamma)$ into the union of the natural image of $\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i}\times_{M_n}Y(\Gamma)$ and the image of the union $\cup_{\Gamma'\in\bar{I}_\Gamma}\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i}\times_{M_n}Y(\Gamma)\cap Y(\Gamma')$. [94]

On the other hand, by using the induced bundle map $\pi_g^*\mathbf{W}^\circ_{canon}|_{Y(\Gamma)\times T(M)}\otimes\mathbf{H}\mapsto\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathbf{H}$ (consult page 21 in section 3) and by using $X\supset Z(s^\circ_{canon})=\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i}$, we realize that the image of the union

$\cup_{\Gamma'\in\bar{I}_\Gamma}\mathcal{M}_{C-\mathbf{M}(E)E-\sum_{e'_i\cdot(C-\mathbf{M}(E)E)<0}e'_i}\times_{M_n}Y(\Gamma)\cap Y(\Gamma')$ in $\mathcal{M}_{C-\mathbf{M}(E)E}\subset X$, the excess component, is nothing but the projection image of the intersection of the section s°_{canon} and the kernel cone, $\pi_{\pi_g^*\mathbf{W}^\circ_{canon}\otimes\mathbf{H}}(s^\circ_{canon}\cap(\cup_{i>0}\mathbf{C}_{\rho_i}))$.[95]

Therefore the blowing ups of these loci into the union of divisors $\cup_{\Gamma'\in\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg}}\hat{D}_{\Gamma'}$ has fitted into the framework of proposition 9 under the identification $\pi_g^*\mathbf{W}^\circ_{canon}|_{X\times_{M_n}Y(\Gamma)}\otimes\mathbf{H}=\mathbf{E}\mapsto\mathbf{F}=\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathbf{H}$ over $\hat{X}_\Gamma\times_{M_n}Y(\Gamma)$.

[94] We have changed the index set of the union of sub-schemes from I_Γ to $\bar{I}_\Gamma$ by remark 17 on page 66.

[95] The kernel cone means the algebraic sub-cone associated to the kernel semi-bundle of the map $\pi_g^*\mathbf{W}^\circ_{canon}\otimes\mathbf{H}\mapsto\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathbf{H}$.

Then by proposition 9, the push-forward of the localized contribution of the top Chern class into $X \times_{M_n} Y(\Gamma)$ under π_h^*

$$\pi_{h*}\{c_{total}(\pi_h^*(\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathbf{H})\otimes_{\Gamma'\in\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg}}\mathcal{O}(-\hat{D}_{\Gamma'}))\cap s_{total}(\pi_r(\hat{D}_\Gamma),\mathbf{C}_{\hat{X}_\Gamma\times_{M_n}Y(\Gamma)}\hat{X}_\Gamma)\}_{dim_{\mathbf{C}}X-rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}}$$

into [96] $\mathcal{A}.(X\times_{M_n}Y(\Gamma))$ is numerically equivalent ($\stackrel{n}{=}$) to the top Chern class of $\pi_g^*\mathbf{W}^\circ_{canon}|_{Y(\Gamma)\times T(M)}\otimes\mathbf{H}$,

By the definition of numerical equivalence (see page 39, definition 5), the push-forward to $\mathcal{A}_0(pt)$ of their pairings to arbitrary complementary dimension cycle classes in $X\times_{M_n}Y(\Gamma)$ are identical. So the original top intersection pairing can be replaced by the much simplified version,

$$= c_{top}(\pi_g^*\mathbf{W}^\circ_{canon}|_{Y(\Gamma)\times T(M)}\otimes\mathbf{H})\cap c_{top}(\pi_g^*\tau_\Gamma\otimes\mathbf{H})\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1},$$

for the purpose of evaluating their push-forward to $\mathcal{A}_0(pt)$.

$\diamondsuit$ To summarize, we have succeeded in casting the original intersection pairing to one on the smooth space $X\times_{M_n}Y(\Gamma)$ which only involves the top Chern classes of $\pi_g^*\mathbf{W}^\circ_{canon}|_{Y(\Gamma)\times T(M)}\otimes\mathbf{H}$ and $\pi_g^*\tau_\Gamma\otimes\mathbf{H}$ and the cycle class $c_1(\mathbf{H})$.

Step Three: Finally we are ready to identify the last expression in Step Two with the mixed algebraic family invariant. Recall the tensor product formula of the top Chern class,

$$c_{top}(\pi_g^*\tau_\Gamma\otimes\mathbf{H}) = \sum_{0\leq t\leq rank_{\mathbf{C}}\tau_\Gamma} c_t(\pi_g^*\tau_\Gamma)\cap c_1(\mathbf{H})^{rank_{\mathbf{C}}\tau_\Gamma-t}.$$

If we insert this identity into the final expression in Step Two, we get

$$= \sum_{0\leq t\leq rank_{\mathbf{C}}\tau_\Gamma} c_{top}(\pi_g^*\mathbf{W}^\circ_{canon}|_{Y(\Gamma)\times T(M)}\otimes\mathbf{H})\cap c_t(\pi_g^*\tau_\Gamma)\cap c_1(\mathbf{H})^{rank_{\mathbf{C}}\tau_\Gamma-t+dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}.$$

Recall that from lemma 6 of [Liu5], $((\Phi_{\mathbf{V}^\circ_{canon}\mathbf{W}^\circ_{canon}},\mathbf{V}^\circ_{canon},\mathbf{W}^\circ_{canon})$ with $\mathbf{V}^\circ_{canon}=\mathbf{V}_{canon}$ is the canonical algebraic family Kuranishi model of the class $C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i$ over the space $M_n\times T(M)$. As we have pointed out on page 81 at the beginning of the current proof that after adding the "correction term" $rank_{\mathbf{C}}\tau_\Gamma$,

$$\begin{aligned} &rank_{\mathbf{C}}\tau_\Gamma + dim_{\mathbf{C}}M_n + rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1 \\ &= dim_{\mathbf{C}}Y(\Gamma) + rank_{\mathbf{C}}(\mathbf{V}^\circ_{canon}-\mathbf{W}^\circ_{canon})+q-1, \end{aligned}$$

is nothing but the expected family dimension of the new class $C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i$ over $Y(\Gamma)\times T(M)$.

[96] Remember that $rank_{\mathbf{C}}\mathbf{W}^\circ_{canon}=rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}$!

Therefore the push-forward of top intersection pairing

$$c_{top}(\pi_g^* \mathbf{W}^{\circ}_{canon}|_{Y(\Gamma)\times T(M)} \otimes \mathbf{H}) \cap c_t(\pi_g^* \tau_\Gamma) \cap c_1(\mathbf{H})^{rank_{\mathbf{C}}\tau_\Gamma - t + dim_{\mathbf{C}} M_n + rank_{\mathbf{C}}(\mathbf{V}_{canon} - \mathbf{W}_{canon}) + q - 1}$$

in $X \times_{M_n} Y(\Gamma) = \mathbf{P}(\mathbf{V}_{canon}) \times_{M_n} Y(\Gamma) = \mathbf{P}(\mathbf{V}^{\circ}_{canon}) \times_{M_n} Y(\Gamma)$ to $\mathcal{A}_0(pt)$ is equal to the mixed algebraic family Seiberg-Witten invariant

$$\mathcal{AFSW}_{M_{n+1}\times_{M_n} Y(\Gamma)\times T(M) \mapsto Y(\Gamma)\times T(M)}(c_t(\tau_\Gamma), C - \mathbf{M}(E)E - \sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i).$$

Thus the total summation over t is

$$\sum_{0 \leq t} \mathcal{AFSW}_{M_{n+1}\times_{M_n} Y(\Gamma)\times T(M) \mapsto Y(\Gamma)\times T(M)}(c_t(\tau_\Gamma), C - \mathbf{M}(E)E - \sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i)$$

$$= \mathcal{AFSW}_{M_{n+1}\times_{M_n} Y(\Gamma)\times T(M) \mapsto Y(\Gamma)\times T(M)}(c_{total}(\tau_\Gamma), C - \mathbf{M}(E)E - \sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i).$$

We are done with the Case I!

Case II: If there exists a type I exceptional class e_{k_i} such that $0 > e_{k_i} \cdot (C - \mathbf{M}(E)E) > e_{k_i}^2$, τ_Γ has been defined to be zero in section 5.2 on page 56. In this case we derive a vanishing result on the top ($= dim_{\mathbf{C}} X - rank_{\mathbf{C}} \mathbf{W}_{canon}$) intersection pairing of $c_1(\mathbf{H})$ with the localized contribution of top Chern class. It is well known that if the total grading of an intersection pairing of characteristic classes exceeds the dimension of the space, the intersection pairing is equal to zero. Our goal is to show that the cap product of the localized contribution of the top Chern class with $c_1(\mathbf{H})^{dim_{\mathbf{C}} M_n + rank_{\mathbf{C}}(\mathbf{V}_{canon} - \mathbf{W}_{canon}) + q - 1}$ vanishes due to dimension count.

For notational simplicity, we assume that $e_{k_1} \cdot (C - \mathbf{M}(E)E) > e_{k_1}^2$. That is to say, we take $i = 1$. Because our argument only makes usage of the dimension count, we do not lose any generality in adopting this convention.

Step One: Firstly we derive a lemma which will be used later.

Lemma 26 *Let $\pi_{\mathbf{F}} : \mathbf{F} \mapsto B$ be a finite rank vector bundle over B. Let $s_{\mathbf{F}}$ denote the zero section embedding $s_{\mathbf{F}} : B \mapsto \mathbf{F}$. Let $r \geq rank_{\mathbf{C}} \mathbf{F}$ be a positive integer. Then for all $\beta \in \mathcal{A}_*(B)$ such that $s_{total}(\mathbf{F}) \cap \beta$ has no grade $< r$ components, the following identity holds,*

$$s_{\mathbf{F}*}\{\beta \cap s_{total}(\mathbf{F})\}_r = \{\pi_{\mathbf{F}}^* \beta\}_r.$$

Proof of the lemma: For all $\alpha \in \mathcal{A}_r(B)$, where r is a fixed natural number $\geq rank_{\mathbf{C}} \mathbf{F}$, we have (see example 3.3.2. on page 67 of [F])

$$s_{\mathbf{F}}^* s_{\mathbf{F}*}\{\alpha\}_r = c_{rank_{\mathbf{C}}\mathbf{F}}(\mathbf{F}) \cap \{\alpha\}_r = \{c_{total}(\mathbf{F}) \cap \alpha\}_{r - rank_{\mathbf{C}}\mathbf{F}}.$$

One can extend this equality trivially to all $\alpha \in \mathcal{A}_{\geq r}(Y(\Gamma_{e_{k_1}}))$ with grading $\geq r$ as they do not contribute to both sides of the identity.

Therefore by the reciprocity property of the total Segre class and the total Chern class and by taking $\alpha = s_{total}(\mathbf{F}) \cap \beta$, we find $s^*_{\mathbf{F}} s_{\mathbf{F}*}\{s_{total}(\mathbf{F}) \cap \beta\}_r = \{\beta\}_{r-rank_{\mathbf{C}}\mathbf{F}}$ for all β satisfying the grading assumption in the lemma.

And therefore $s_{\mathbf{F}*}\{s_{total}(\mathbf{F}) \cap \beta\}_r = \{\pi^*_{\mathbf{F}}\beta\}_r$ because the Gysin homomorphism satisfies $s^*_{\mathbf{F}} = (\pi^*_{\mathbf{F}})^{-1}$ (please consult page 65, definition 3.3. of [F]). The lemma is proved. $\Box$

Step Two: The yet-to-be-identified intersection pairing

$$\{ \sum_{i=1}^{i=rank_{\mathbf{C}}\mathbf{W}_{canon}} (-1)^{i-1} c_{rank_{\mathbf{C}}\mathbf{W}_{canon}-i}(\pi^*_{\tilde{\hat{X}}_\Gamma}\mathbf{W}_{canon} \otimes \mathbf{H} \otimes_{\Gamma' \in \bar{I}_\Gamma - \bar{I}^{\gg}_\Gamma} \mathcal{O}(-\hat{D}_{\Gamma'}))$$

$$\cap \hat{D}^{i-1}_\Gamma \cap [\hat{D}_\Gamma] \cap c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n + rank_{\mathbf{C}}(\mathbf{V}_{canon} - \mathbf{W}_{canon}) + q - 1}\}$$

can be pushed-forward as a zero dimensional cycle class into $\mathcal{A}_0(\hat{X}_\Gamma \times_{M_n} Y(\Gamma))$. Because that $Y(\Gamma) \subset Y(\Gamma_{e_{k_1}})$, $\hat{X}_\Gamma \times_{M_n} Y(\Gamma) \subset \hat{X}_\Gamma \times_{M_n} Y(\Gamma_{e_{k_1}})$. Similar to what was done in step one of Case I, we may deform to the projectified normal cone and replace $\hat{X}_\Gamma \times_{M_n} Y(\Gamma) \subset \hat{X}_\Gamma$ by the inclusion into the zero section of the projectified normal cone of $\hat{X} \times_{M_n} Y(\Gamma_{e_{k_1}})$,

$$\hat{X}_\Gamma \times_{M_n} Y(\Gamma) \subset \hat{X}_\Gamma \times_{M_n} Y(\Gamma_{e_{k_1}}) \subset \mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma \times_{M_n} Y(\Gamma_{e_{k_1}})} \hat{X}_\Gamma \oplus 1).$$

Correspondingly, we twist the obstruction vector bundle $\pi^*_f \pi^*_g \mathbf{W}_{canon}$ by [97] $\mathcal{O}(\hat{\mathbf{P}}_\infty)$.

Then the derived exact sequence of locally free sheaves

$$\mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{M}(E)E+\Xi_{k_1}} \otimes \mathcal{E}_C\big) \mapsto \mathcal{R}^0\pi_*\big(\mathcal{O}_{\mathbf{M}(E)E} \otimes \mathcal{E}_C\big) \mapsto \mathcal{R}^1\pi_*\big(\mathcal{O}_{\Xi_{k_1}} \otimes \mathcal{E}_{C-\mathbf{M}(E)E}\big)$$

induces a short exact sequence analogous to the short exact sequence on page 21. One can interpret the construction of this new sequence as a special case of the general construction once we "formally" consider e_{k_1} to be the unique type I exceptional class which pairs negatively with $C - \mathbf{M}(E)E$. For notational simplicity, we still denote the corresponding sequence of vector bundles by the same notation [98] as before, $0 \mapsto \underline{\mathbf{W}}_{canon} \mapsto \mathbf{W}_{canon} \mapsto \mathbf{V}_{quot} \mapsto 0$. This sequence breaks $\mathbf{W}_{canon}|_{Y(\Gamma)\times T(M)}$ into the factors $\underline{\mathbf{W}}_{canon}$ and $\mathbf{V}_{quot}$. In the current context, the symbol $\mathbf{V}_{quot}$ means the vector bundle associated with the locally free summand of $\mathcal{R}^1\pi_*\big(\mathcal{O}_{\Xi_{k_1}} \otimes \mathcal{E}_{C-\mathbf{M}(E)E}\big)$. Then by proposition 10, we may identify the above expression with

[97] The twisting of this line bundle $\mathcal{O}(\hat{\mathbf{P}}_\infty)$ does not play an essential role in our argument, its presence only makes the notations slightly more complicated. Nevertheless we do not remove it in Case II as we do not always keep our cycle disjoint from $\hat{\mathbf{P}}_\infty$.

[98] In our current argument only the ranks of these bundles matter. As far as we do not use specific properties of these bundles, the slight abuse of notations does not cause trouble.

$$\{c_{total}((\pi_g\pi_f)^*(\underline{\mathbf{W}}_{canon}\otimes\mathcal{O}(\hat{\mathbf{P}}_\infty)\otimes\mathbf{H})\otimes_{\Gamma'\in\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg}}\mathcal{O}(-\pi_f^*\hat{D}_{\Gamma'}))\cap s_{total}(\pi_r(\hat{D}_\Gamma),\mathbf{C}_{\hat{X}_\Gamma\times_{M_n}Y(\Gamma)}\hat{X}_\Gamma)\}_{dim_{\mathbf{C}}X-\underline{\mathbf{W}}_{canon}}$$

$$\cap c_{top}(\pi_f^*(\pi_g^*\mathbf{V}_{quot}\otimes\mathcal{O}(\hat{\mathbf{P}}_\infty)\otimes\mathbf{H}))\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}.$$

Now we push forward this grade zero cycle class into $\mathcal{A}_0(\mathbf{P}(\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X\oplus 1))$ by [99] $\pi_{\mathbf{P}*}$. Take $\mathbf{P}_\infty=\mathbf{P}(\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X)$ to be the divisor at infinity in $\mathbf{P}(\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X\oplus 1)$. By proposition 8 and the observation/argument used in proposition 16, the $\pi_{\mathbf{P}*}$ push-forward of the localized contribution of top Chern class $\{c_{total}((\pi_g\pi_f)^*(\underline{\mathbf{W}}_{canon}\otimes\mathcal{O}(\mathbf{P}_\infty)\otimes\mathbf{H})\otimes_{\Gamma'\in\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg}}\mathcal{O}(-\pi_f^*\hat{D}_{\Gamma'}))\cap s(\pi_r(\hat{D}_\Gamma),\mathbf{C}_{\hat{X}_\Gamma\times_{M_n}Y(\Gamma)}\hat{X}_\Gamma)\}_{dim_{\mathbf{C}}X-\underline{\mathbf{W}}_{canon}}$ can be written as the difference of two localized contributions of top Chern class,

$$\{c_{total}(\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathcal{O}(\mathbf{P}_\infty)\otimes\mathbf{H})\cap s_{total}(Z(\underline{s}_{canon})\times_{M_n}Y(\Gamma),\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X)$$

$$-c_{total}(\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathcal{O}(\mathbf{P}_\infty)\otimes\mathbf{H})\cap s_{total}(\cup_{\Gamma'\in\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg}}Z(\underline{s}_{canon})\times_{M_n}(Y(\Gamma')\cap Y(\Gamma)),\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X)\}_{dim_{\mathbf{C}}X-\underline{\mathbf{W}}_{canon}},$$

where the section $\underline{s}_{canon}$ is the induced section of $\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathbf{H}$ by

$$\pi_X^*\mathbf{W}^\circ_{canon}|_{X\times_{M_n}Y(\Gamma)}\mapsto\underline{\mathbf{W}}_{canon}$$

and s°_{canon} and the sub-schemes $Z(\underline{s}_{canon})\times_{M_n}Y(\Gamma)$ and $\cup_{\Gamma'\in\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg}}Z(\underline{s}_{canon})\times_{M_n}(Y(\Gamma')\cap Y(\Gamma))$ are embedded in the zero cross section $X\times_{M_n}Y(\Gamma_{e_{k_1}})\subset\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X$.

Set $Z_1=Z(\underline{s}_{canon})\times_{M_n}Y(\Gamma)$ and $Z_2=\cup_{\Gamma'\in\bar{I}_\Gamma-\bar{I}_\Gamma^{\gg}}Z(\underline{s}_{canon})\times_{M_n}(Y(\Gamma')\cap Y(\Gamma))$ and $i_{Z_a}:Z_a\subset\mathbf{P}(\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X\oplus 1)$ for $a=1,2$.

We will give a uniform argument for both $a=1,2$ that the top intersection pairings

$$i_{Z_a*}\{c_{total}(\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathcal{O}(\mathbf{P}_\infty)\otimes\mathbf{H})\cap s_{total}(Z_a,\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X)\}_{dim_{\mathbf{C}}X-rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}}$$

$$\cap c_{top}(\pi^*_{\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X}\pi_g^*\mathbf{V}_{quot}\otimes\mathcal{O}(\mathbf{P}_\infty)\otimes\mathbf{H})\cap c_1(\pi^*_{\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X}\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}$$

vanish identically.

Step Three: Because $e_{k_1}^2<e_{k_1}\cdot(C-\mathbf{M}(E)E)$, one observes that the expected dimension of the family moduli space of $C-\mathbf{M}(E)E-e_{k_1}$ over $Y(\Gamma_{e_{k_1}})$,

$$dim_{\mathbf{C}}Y(\Gamma_{e_{k_1}})+p_g+\frac{(C-\mathbf{M}(E)E-e_{k_1})^2-c_1(\mathbf{K}_{M_{n+1}/M_n})\cdot(C-\mathbf{M}(E)E-e_{k_1})}{2}$$

[99]Here $\pi_{\mathbf{P}}:\mathbf{P}(\mathbf{C}_{\hat{X}_\Gamma\times_{M_n}Y(\Gamma_{e_{k_1}})}\hat{X}\oplus 1)\mapsto\mathbf{P}(\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X\oplus 1)$ is the projection map.

$$= dim_{\mathbf{C}}M_n+p_g+\frac{e_{k_1}^2-c_1(\mathbf{K}_{M_{n+1}/M_n})\cdot e_{k_1}}{2}+\frac{(C-\mathbf{M}(E)E-e_{k_1})^2-c_1(\mathbf{K}_{M_{n+1}/M_n})\cdot(C-\mathbf{M}(E)E-e_{k_1})}{2}$$

$$< dim_{\mathbf{C}}M_n+p_g+\frac{(C-\mathbf{M}(E)E)^2-c_1(\mathbf{K}_{M_{n+1}/M_n})\cdot(C-\mathbf{M}(E)E)}{2},$$

strictly smaller than the expected family dimension of the class $C-\mathbf{M}(E)E$ over the family M_n. We use this observation to derive the vanishing result.

Define $B = X\times_{M_n} Y(\Gamma_{e_{k_1}})$, $\mathbf{F}=\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X$, $\iota_{\mathbf{F}}:\mathbf{F}\subset \mathbf{P}(\mathbf{F}\oplus 1)$ and $j_{Z_a}: Z_a\subset X\times_{M_n}Y(\Gamma_{e_{k_1}})=B$. Then i_{Z_a} can be factorized as $\iota_{\mathbf{F}}\circ s_{\mathbf{F}}\circ j_{Z_a}$. Because both $Z_a\subset X\times_{M_n}Y(\Gamma)\subset X\times_{M_n}Y(\Gamma_{e_{k_1}})$, there is a short exact sequence of normal cones (see example 4.1.6. of [F] for its definition),

$$0\mapsto j_{Z_a}^*\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X\mapsto \mathbf{C}_{Z_a}\mathbf{N}_{X\times_{M_n}Y(\Gamma_{e_{k_1}})}X\mapsto \mathbf{C}_{Z_a}X\times_{M_n}Y(\Gamma_{e_{k_1}})\mapsto 0.$$

By the product property of the total Segre classes for short exact sequences of cones, the final expression in Step Two can be re-casted into

$$\iota_{\mathbf{F}*}s_{\mathbf{F}*}j_{Z_a*}\{c_{total}(\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathcal{O}(\mathbf{P}_\infty)\otimes\mathbf{H})\cap s_{total}(Z_a,X\times_{M_n}Y(\Gamma_{e_{k_1}}))\cap s_{total}(j_{Z_a}^*\mathbf{F})\}_{dim_{\mathbf{C}}X-rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}}$$

$$\cap c_{top}(\pi_{\mathbf{P}(\mathbf{F}\oplus 1)}^*(\pi_g^*\mathbf{V}_{quot}\otimes\mathbf{H})\otimes\mathcal{O}(\mathbf{P}_\infty))\cap c_1(\pi_{\mathbf{P}(\mathbf{F}\oplus 1)}^*\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}$$

$$=\iota_{\mathbf{F}*}\{s_{\mathbf{F}*}\{j_{Z_a*}(c_{total}(\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathbf{H})\cap s_{total}(Z_a,X\times_{M_n}Y(\Gamma_{e_{k_1}})))\cap s_{total}(\mathbf{F})\}_{dim_{\mathbf{X}}-rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}}$$

$$\cap c_{top}(\pi_{\mathbf{F}}^*(\pi_g^*\mathbf{V}_{quot}\otimes\mathbf{H})\otimes\mathcal{O}(\mathbf{P}_\infty))\cap c_1(\pi_{\mathbf{F}}^*\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}\}.$$

Define $\beta=j_{Z_a*}(c_{total}(\pi_g^*\underline{\mathbf{W}}_{canon}\otimes\mathbf{H})\cap s_{total}(Z_a,X\times_{M_n}Y(\Gamma_{e_{k_1}})))\in\mathcal{A}.(X\times_{M_n}Y(\Gamma_{e_{k_1}}))$ and set $r=dim_{\mathbf{C}}X-rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}$. Then $\{\beta\cap s_{total}(\mathbf{F})\}_r$ is the push-forward of the localized contribution of top Chern class over $Z_a\subset\mathbf{F}$ into $\mathcal{A}_r(X\times_{M_n}Y(\Gamma_{e_{k_1}}))$. Then by e.g. proposition 13 of [100] [Liu5], $\{\beta\cap s_{total}(\mathbf{F})\}_s=0$ for all $s<r$. Namely, the localized contribution of top Chern class is the leading (lowest grading) term of the intersection pairing.

Then the assumption of lemma 26 of Step One is applicable and we know $s_{\mathbf{F}*}\{\beta\cap s_{total}(\mathbf{F})\}_r=\{\pi_{\mathbf{F}}^*\beta\}_r$. Because the bundle projection $\pi_{\mathbf{F}}:\mathbf{F}\mapsto B=X\times_{M_n}Y(\Gamma_{e_{k_1}})$ is flat of relative dimension $rank_{\mathbf{C}}\mathbf{F}$, and the inclusion $\iota_{\mathbf{F}}$ is a proper morphism, and by theorem 3.2.(c)-(d). on pages 50-51 of [F], and the fact that the flat pull-back $\pi_{\mathbf{F}}^*:\mathcal{A}_{r-rank_{\mathbf{C}}\mathbf{F}}(B)\mapsto\mathcal{A}_r(\mathbf{F})$ lifts the gradings up by $rank_{\mathbf{C}}\mathbf{F}$, we may rewrite the above intersection pairing as [101]

$$\iota_{\mathbf{F}*}\{\pi_{\mathbf{F}}^*\{\beta\}_{dim_{\mathbf{C}}X-rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}-rank_{\mathbf{C}}\mathbf{F}}\cap c_1(\pi_{\mathbf{F}}^*\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}\cap c_{top}(\iota_{\mathbf{F}}^*(\pi_{\mathbf{P}(\mathbf{F}\oplus 1)}^*(\pi_g^*\mathbf{V}_{quot}\otimes\mathbf{H})\otimes\mathcal{O}(\mathbf{P}_\infty)))\}$$

[100] We had used this fact earlier in the proof of proposition 10, too.

[101] After we introduce $\pi_{\mathbf{F}}^*$ into our formulae, the cycle class is not "refined" in $X\times_{M_n}Y(\Gamma_{e_{k_1}})$ any more!

$$= \iota_{\mathbf{F}*}\{\pi^*_{\mathbf{F}}(\{\beta\}_{dim_{\mathbf{C}}X-rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}-rank_{\mathbf{C}}\mathbf{F}}\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1})\cap c_{top}(\iota^*_{\mathbf{F}}\pi^*_{\mathbf{P}(\mathbf{F}\oplus 1)}(\pi^*_g\mathbf{V}_{quot}\otimes\mathbf{H})\otimes\mathcal{O}(\mathbf{P}_\infty))\}$$

$$= \iota_{\mathbf{F}*}\{\pi^*_{\mathbf{F}}(\{\beta\}_{dim_{\mathbf{C}}Y(\Gamma_{e_{k_1}})-rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}}\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}X-rank_{\mathbf{C}}\mathbf{W}_{canon}})\cap c_{top}(i^*_{\mathbf{F}}\pi^*_{\mathbf{P}(\mathbf{F}\oplus 1)}(\pi^*_g\mathbf{V}_{quot}\otimes\mathbf{H})\otimes\mathcal{O}(\mathbf{P}_\infty))\}$$

$$= \iota_{\mathbf{F}*}\{\pi^*_{\mathbf{F}}\{\beta\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}X-rank_{\mathbf{C}}\mathbf{W}_{canon}}\}_{dim_{\mathbf{C}}Y(\Gamma_{e_{k_1}})-rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}-(dim_{\mathbf{C}}X-rank_{\mathbf{C}}\mathbf{W}_{canon})}\}\cap c_{top}(\pi^*_{\mathbf{P}(\mathbf{F}\oplus 1)}(\pi^*_g\mathbf{V}_{quot}\otimes\mathbf{H})\otimes\mathcal{O}(\mathbf{P}_\infty)).$$

We have used $dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{F} = dim_{\mathbf{C}}Y(\Gamma_{e_{k_1}})$ in the above derivation.

Yet the grading $dim_{\mathbf{C}}Y(\Gamma_{e_{k_1}}) - rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon} - (dim_{\mathbf{C}}X - rank_{\mathbf{C}}\mathbf{W}_{canon})$ of $\{\bullet\}$ is exactly the difference between the expected family dimension of the class $C - \mathbf{M}(E)E - e_{k_1}$ over $Y(\Gamma_{e_{k_1}})$ and [102] the expected family dimension of the class $C - \mathbf{M}(E)E$ over M_n.

As we assume $e^2_{k_1} < e_{k_1} \cdot (C - \mathbf{M}(E)E) < 0$, we have already shown at the beginning of Step Three that this grading is negative. Therefore

$$\pi^*_{\mathbf{F}}\{\beta\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}X-rank_{\mathbf{C}}\mathbf{W}_{canon}}\}_{\mathbf{C}Y(\Gamma_{e_{k_1}})-rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}-(dim_{\mathbf{C}}X-rank_{\mathbf{C}}\mathbf{W}_{canon})} = 0$$

and therefore the whole intersection pairing vanishes. In particular its push-forward into $\mathcal{A}_0(pt) \cong \mathbf{Z}$ is zero. As this holds for both Z_1 and Z_2, the original intersection pairing (their difference) is also zero. We are done with Case II.

As we have finished the identification with the mixed algebraic family Seiberg-Witten invariants in both cases, we have finished the proof of proposition 18.

□

Remark 19 *In the proof of the proposition 18, we only discuss the non-restricted case. If one specifies a point $t_L \in T(M)$ and would like to count curves in $\mathcal{M}_{C-\mathbf{M}(E)E}$ whose images in M are in the linear system $|L|$ specified by the point t_L, there are two viewpoints one can adopt.*

(i). By restricting to a point $t_L \in T(M)$, effectively one shrinks $T(M)$ to a point. One can think of this procedure as a formal reduction of the irregularity $q \to 0$ and the rest of the deduction is identical to the $q = 0$ case, where $T(M)$ does not play any role here.

(ii). Alternatively, one may replace the family moduli space $\mathcal{M}_{C-\mathbf{M}(E)E}$, the total projective space bundle $X = \mathbf{P}(\mathbf{V}_{canon})$, etc., by their t_L-restricted counterparts, $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{T(M)} \{t_L\}$ and $X \times_{T(M)} \{t_L\}$, respectively. One may insert the cycle class $[t_L] \in \mathcal{A}_0(T(M))$ into the intersection theory product and therefore replace the power $c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})+q-1}$ by $[t_L] \cap c_1(\mathbf{H})^{dim_{\mathbf{C}}M_n+rank_{\mathbf{C}}(\mathbf{V}_{canon}-\mathbf{W}_{canon})-1}$ and the rest of the discussion goes through without any change. By either angles the reader should be able to make

[102] We have used the observation that $rank_{\mathbf{C}}\mathbf{W}^\circ_{canon} = rank_{\mathbf{C}}\underline{\mathbf{W}}_{canon}$ implicitly.

the suitable adjustments in all the formulae and finish the proof. We do not repeat the redundant details here [103].

6.3 The Proof of the Main Theorem

We are ready to combine all the results proved in the paper to prove the main theorem of the paper,

Theorem 1 *Let $\delta \in \mathbf{N}$ denote* [104] *the number of nodal singularities. Let L be a $5\delta - 1$ very-ample line bundle on an algebraic surface M, then the number of δ nodes nodal singular curves in a generic δ dimensional linear sub-system of $|L|$ can be expressed as a universal polynomial (independent to M) of $c_1(L)^2$, $c_1(L) \cdot c_1(M)$, $c_1(M)^2$, $c_2(M)$ of degree δ.*

For the invertible sheaf $\mathcal{L} = \mathcal{E}_C \mapsto M \times T(M)$ parametrized by a cohomology class $C \in H^{1,1}(M, \mathbf{Z})$, we may extend the definition of $k-$very ampleness by assuming the surjectivity of the restriction morphism $\mathcal{R}^0\pi_{T(M)*}(\mathcal{L}) \mapsto \mathcal{R}^0\pi_{T(M)*}(\mathcal{L} \otimes \mathcal{O}_{Z \times T(M)})$ for all length $k+1$ sub-schemes $Z \subset M$.

Remark 20 *Let $\delta \in \mathbf{N}$ denote the number of nodal singularities. Let C be a cohomology class in $H^{1,1}(M, \mathbf{Z})$ and let $\mathcal{L} \mapsto M \times T(M)$ be the invertible sheaf with $c_1(i_M^*\mathcal{L}) = C$, where $i_M : M \times \{0\} \subset M \times T(M)$. Suppose that $\mathcal{L}$ is $5\delta - 1-$very ample and one can find generic δ dimensional non-linear sub-system of the projective space bundle $\mathbf{P}(\pi_{T(M)*}(\mathcal{L})^*) \cong \mathbf{P}(\mathbf{V}_{canon})$, then one may formulate a corresponding theorem for $\mathcal{L}$, parallel to theorem 1. The universal polynomial associated to $\mathcal{L}$ is the product of the universal polynomial found in theorem 1 and*[105] *$\mathcal{ASW}(C)$.*

Proof of the main theorem: Let L be a line bundle over M with $c_1(L) = C$, then L determines a unique point $t_L \in T(M)$ in the connected component of the Picard variety. As usual $T(M)$ represents the component of Picard group of M parametrizing the line bundles with first Chern class C. Let $m_1 = m_2 = \cdots = m_\delta = 2$ and let $\mathcal{M}_{C-\mathbf{M}(E)E}$ denote the algebraic family moduli space of curves dual to $C - 2\sum_{i \leq \delta} E_i$ which projects to $M_\delta \times T(M)$. Then $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{T(M)} \{t_L\}$ is the sub-moduli space of curves whose projection into M lie in $|L|$. Let $\mathcal{M}_V$ denote the pre-image of V, a general δ dimensional linear subsystem of $|L|$ under the projection map $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{T(M)} \{t_L\} \mapsto |L|$, then $\mathcal{M}_V$ can be viewed as the $\frac{L^2 - \mathbf{K}_M \cdot L}{2} - q(M) + p_g(M) - \delta$-fold generic hyperplane intersection of $|L| = \mathbf{P}(\mathbf{V}_{canon}) \times_{T(M)} \{t_L\}$, intersecting with the family moduli space of $C-$

[103] Please compare with remark 16, located right after the definition of the modified algebraic family invariants.
[104] In the proof of the main theorem, we switch from n to δ, fitting to Göttsche's convention.
[105] Refer to remark 15.

$\mathbf{M}(E)E$, $\mathcal{M}_{C-\mathbf{M}(E)E}$. It can be also viewed as the set theoretical intersection resembling the following intersection theoretical product,

$$c_{top}(\pi_X^*\mathbf{W}_{canon}\otimes\mathbf{H})\cap c_1(\mathbf{H})^{dim_{\mathbf{C}}M_\delta+\frac{L^2-\mathbf{K}_M\cdot L}{2}-q(M)+p_g(M)-3\delta}\cap[X\times_{T(M)}t_L]$$

where $\mathcal{M}_{C-\mathbf{M}(E)E}$ is represented by the top Chern class of the canonical algebraic obstruction vector bundle $\pi_X^*\mathbf{W}_{canon}\otimes\mathbf{H}$ over $X=\mathbf{P}(\mathbf{V}_{canon})$ and $[X\times_{T(M)}t_L]\in\mathcal{A}_{dim_{\mathbf{C}}X-q}(X)$ is the fiber cycle class determined by the point t_L.

This object is nothing but the mixed algebraic family Seiberg-Witten invariant of $C-\sum 2E_i$, with an additional $[t_L]$ inserted, to restrict $\mathcal{L}$ to L, i.e. $\mathcal{AFSW}_{M_{\delta+1}\times\{t_L\}\mapsto M_\delta\times\{t_L\}}(1,C-2\sum_{1\leq i\leq\delta}E_i)$.

By the discussion presented in subsection 6.4 below, if we choose the linear subsystem V generically, then the set $\mathcal{M}_V$ can be decomposed into a portion over S_{γ_δ}, $\mathcal{M}_V\times_{M_\delta}(M_\delta-\cup_{\Gamma\in\Delta(\delta)-\{\gamma_\delta\}}Y_\Gamma)$ and the excess component $\mathcal{M}_V\times_{M_\delta}(\cup_{\Gamma\in\Delta(\delta)-\{\gamma_\delta\}}Y_\Gamma)$.

By proposition 21 the $5\delta-1$-very ampleness condition on L implies that $\mathcal{M}_V\times_{M_\delta}(M_\delta-\cup_{\Gamma\in\Delta(\delta)-\{\gamma_\delta\}}Y_\Gamma)$ has the structure of a finite scheme which maps into the generic stratum Y_{γ_δ}. I.e. its image will miss all those Y_Γ associated with fan-like $\Gamma\in adm_2(\delta)$. In particular, both $\mathcal{M}_V\times_{M_\delta}(M_\delta-\cup_{\Gamma\in\Delta(\delta)}Y_\Gamma)$ and $\mathcal{M}_V\times_{M_\delta}(\cup_{\Gamma\in\Delta(\delta)}Y_\Gamma)$ are closed sub-schemes of X.

We emphasize that we do **NOT** use the very ampleness condition on L to gain any regularity of the sub-scheme $\mathcal{M}_V\times_{M_\delta}(\cup_{\Gamma\in\Delta(\delta)-\{\gamma_\delta\}}Y_\Gamma)$. Instead, the machineries developed earlier in this paper, namely residual intersection formula of top Chern classes, recursive blowing ups of $X=\mathbf{P}(\mathbf{V}_{canon})$ in subsection 5.1 and proposition 18, remark 19, etc., allows us to identify through an induction argument the intersection numbers represented by $\mathcal{M}_V\times_{M_\delta}Y_{\gamma_\delta}$, i.e. the top intersection pairing of the push-forward of the localized top Chern class of $\pi_X^*\mathbf{W}_{canon}\otimes\mathbf{H}$ over $\mathcal{M}_V\times_{M_\delta}Y_{\gamma_\delta}$ with a complementary power of $c_1(\mathbf{H})$, with the modified family invariant $\mathcal{AFSW}^*$ of $C-\mathbf{M}(E)E$ over $M_\delta\times\{t_L\}$, namely the difference of $\mathcal{AFSW}_{M_{\delta+1}\times\{t_L\}\mapsto M_\delta\times\{t_L\}}(1,C-\mathbf{M}(E)E)$ and the sum of a hierarchy of the modified mixed algebraic family Seiberg-Witten invariants of $C-\mathbf{M}(E)E-\sum_{e_i\cdot(C-\mathbf{M}(E)E)<0}e_i$ above $Y(\Gamma)$, for various $\Gamma\in\Delta(\delta)-\{\gamma_\delta\}$. Thus we may identify the degree of the finite cycle class $[\mathcal{M}_V\times_{M_\delta}(M_\delta-\cup_{\Gamma\in\Delta(\delta)-\{\gamma_\delta\}}Y_\Gamma)]$ in $\mathcal{A}_0(pt)\cong\mathbf{Z}$ with the modified algebraic family Seiberg-Witten invariant, $\mathcal{AFSW}^*_{M_{\delta+1}\times T(M)\mapsto M_\delta\times T(M)}(\{t_L\},C-2\sum_{1\leq i\leq\delta}E_i)$, defined following remark 12 and parallel to definition 14.

According to proposition 13, remark 16 and remark 19, this modified family invariant can be expressed as $\mathcal{ASW}(\{t_L\},C)$ times a universal degree δ polynomial of $C^2, C\cdot c_1(M), c_1(M)^2$ and $c_2(M)$. Because $\mathcal{ASW}(\{t_L\},C)=c_1(\mathbf{H})^{p_g-q+\frac{C^2-c_1(\mathbf{K}_M)\cdot C}{2}}[\mathbf{P}(H^0(M,L))]\cong 1$ and $C^2=L\cdot L, C\cdot c_1(M)=-L\cdot\mathbf{K}_M$, $c_1(M)^2=\mathbf{K}_M\cdot\mathbf{K}_M$, the integer can be expressed as a universal polynomial of $L^2, L\cdot\mathbf{K}_M, \mathbf{K}_M\cdot\mathbf{K}_M$ and $c_2(M)$.

On the other hand, the symmetric group of δ elements, $\mathbf{S}_\delta$, acts naturally and freely upon the open stratum Y_{γ_δ}, whose underlying set is the set of all

ordered distinct δ-tuples of points in M. This free action induces free actions upon the sub-scheme $\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta}$ and the smooth ambient space $X \times_{M_\delta} Y_{\gamma_\delta}$. We have the following proposition, whose proof will be postponed after we have finished the proof of our main theorem.

Proposition 19 *Assuming that $\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta} \subset V \times M_\delta \times \{t_L\}$ ($\subset X$) is a finite sub-scheme. Then the push-forward of the zero cycle $[\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta}] \in \mathcal{A}_0(X)$ into $\mathcal{A}_0(pt)$ is equal to $\delta!$ times "the number of δ-node nodal curves", $d_\delta(L)$, defined [106] by Göttsche [Got].*

When L is $3\delta-1$-very ample, the sub-scheme $W \subset V \times M_{2,0}^{\delta}$ in the proof of proposition 20 is a finite scheme for a generic choice of δ dimensional linear-sub-system $V \subset |L|$. Under such an assumption, $\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta}$ is a finite scheme as well. From proposition 19, we know that the degree of $[\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta}]$ is equal to $\delta! \cdot d_\delta(L)$. As we have assumed that L is $5\delta-1$-very ample, Göttsche (in proposition 20) has shown that $d_\delta(L)$ actually represents the number of $\delta-$node nodal singular curves in a generic δ dimensional linear-subsystem V.

As we have identified the degree of $[\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta}]$ by two different ways, we find that the number of δ-nodes nodal curves in a generic δ dimensional $V \subset |L|$ (counted with multiplicities), $d_\delta(L)$, is equal to $\frac{1}{\delta!}\mathcal{AFSW}^*_{M_{\delta+1}\times\{t_L\}\mapsto M_\delta\times\{t_L\}}(1, C-2\sum_{1\leq i\leq \delta} E_i)$. By proposition 13 and its ending remark 16 it is a universal degree δ polynomial in terms of the four variables L^2, $L\cdot\mathbf{K}_M$, $\mathbf{K}_M\cdot\mathbf{K}_M$ and $c_2(M)$. So we have finished the proof of our main theorem. $\square$

Remark 21 *If we replace the singular multiplicities 2 by $m_1 = m_2 = \cdots = m_\delta = m > 2$, and replace the $5\delta-1-$very-ampleness condition on $L \mapsto M$ by an $(\frac{(m+1)(m+2)}{2}-1)\delta-1-$very-ampleness condition, our main theorem can be generalized to count curves with δ ordinary multiplicities m singularities [107]. And the argument is completely parallel to the above argument.*

At the end of this subsection, we offer a proof of proposition 19 cited above. Proof of proposition 19: We observe that the canonical algebraic obstruction vector bundle $\pi_X^*\mathbf{W}_{canon}\otimes\mathbf{H}$ of the class $C-\mathbf{M}(E)E$ restricts to an $\mathbf{S}_\delta$ invariant vector bundle over $X\times_{M_\delta\times T(M)}(Y_{\gamma_\delta}\times\{t_L\})$. This is easy to check by using the definition of $\mathbf{W}_{canon}$ (see section 5.1, definition 5.3 of [Liu3]) and the fact that different exceptional $\mathbf{CP}^1$s are completely disjoint and are permuted transitively under an induced $\mathbf{S}_\delta$ action. Then $\pi_X^*\mathbf{W}_{canon}\otimes\mathbf{H}$ descends to a vector bundle on the free quotient $(X\times_{M_\delta} Y_{\gamma_\delta})/\mathbf{S}_\delta\times_{T(M)}\{t_L\}$, denoted by $\mathbf{W}_{descend}$.

As usual let $M^{[3\delta]}$ denote the Hilbert scheme of M parametrizing the length 3δ sub-schemes of M. Consider the universal sub-scheme $Z_{3\delta}(M)\subset M\times M^{[3\delta]}$ and the projection maps,

[106] Consult the discussion in next subsection.

[107] in a general $(\frac{m(m+1)}{2}-2)\delta$ dimensional linear sub-system of $|L|$.

$$\begin{array}{ccc} Z_{3\delta}(M) & \stackrel{q_{3\delta}}{\longrightarrow} & M^{[3\delta]} \\ \Big\downarrow{\scriptstyle p_{3\delta}} & & \\ M & & \end{array}$$

The fibration of universal divisors (curves) $D \mapsto |L|$ of the linear system forms a divisor in $M \times |L|$, and is called the universal divisor. The line bundle $\mathcal{O}_{|L|\times M}(D)$ for the universal divisor $D \subset |L| \times M$ is equivalent to $\pi^*_{|L|\times M \mapsto M} L \otimes \pi^*_{|L|\times M\mapsto |L|}\mathbf{H}$, twisted by the hyperplane line bundle $\mathbf{H} \mapsto |L|$. Then we may consider $q_{3\delta *}p^*_{3\delta}(L\otimes \mathbf{H}) = \mathbf{H}\otimes q_{3\delta *}p^*_{3\delta}L = \mathbf{H}\otimes L_{3\delta}$, of rank 3δ over $|L| \times M^{[3\delta]}$.

The smooth quotient space $Y_{\gamma_\delta}/\mathbf{S}_\delta$ parametrizes the un-ordered δ-tuples of distinct points in M and is embedded naturally onto the top open stratum of $M^{[\delta]}$. On the other hand, let $x_1, x_2, \cdots, x_\delta$ be distinct δ points on M. Then $\coprod_{1\leq i\leq \delta} Spec(\mathcal{O}_{M,x_i}/m^2_{M,x_i})$ is a length 3δ sub-scheme of M. This enables us to embed the top open stratum of $M^{[\delta]}$ into $M^{[3\delta]}$. Denote this composite inclusion by $\Pi_\delta : Y_{\gamma_\delta}/\mathbf{S}_\delta \mapsto M^{[3\delta]}$. Then we first notice that $\Pi^*_\delta(L_{3\delta}\otimes \mathbf{H}) = \mathbf{W}_{descend}$, i.e. the descend of our canonical algebraic family obstruction bundle coincides with the obstruction bundle defined by Göttsche [108] when they are both restricted to the top open strata. Again it is because when the δ blowing up points are distinct in M, the corresponding exceptional divisors $E_i \subset M_{\delta+1}\times_{M_\delta} Y(\gamma_\delta)$, $1 \leq i \leq \delta$, are all disjoint.

Then we have the following short exact sequence [109]

$$0 \mapsto \mathcal{O}_{M_{\delta+1}\times_{M_\delta} Y_{\gamma_\delta}\times\{t_L\}}(-2\sum_{1\leq i\leq\delta} E_i)\otimes f^*_{\delta,1}\mathcal{L} \mapsto \mathcal{O}_{M_{\delta+1}\times_{M_\delta} Y_{\gamma_\delta}\times\{t_L\}}\otimes f^*_{\delta,1}\mathcal{L} \mapsto \mathcal{O}_{2\sum_{1\leq i\leq\delta} E_i|_{M_{\delta+1}\times_{M_\delta} Y_{\gamma_\delta}}}\otimes f^*_{\delta,1}\mathcal{L} \mapsto 0,$$

which is the fundamental building block of the t_L-restricted version of the canonical algebraic family Kuranishi model of $C - \mathbf{M}(E)E$.

The push-forward of $\mathcal{O}_{M_{\delta+1}\times_{M_\delta} Y_{\gamma_\delta}}(-2\sum_{1\leq i\leq\delta} E_i) \subset \mathcal{O}_{M_{\delta+1}\times_{M_\delta} Y_{\gamma_\delta}}$ to $M \times Y_{\gamma_\delta}$ defines an ideal sheaf of a universal sub-scheme. It is invariant under a free $\mathbf{S}_\delta$ action and we denote its free quotient under $\mathbf{S}_\delta$ by the new notation Z_{γ_δ}.

On the other hand $M_{\delta+1} \mapsto M \times M_\delta$ projects to the trivial bundle $M \times M_\delta$ over M_δ. So the $M_{\delta+1}\times_{M_\delta} Y_{\gamma_\delta} \longrightarrow M \times Y_{\gamma_\delta}$-push-forward of the above short exact sheaf sequence results in an $\mathbf{S}_\delta$ invariant short exact sequence which descends to a short exact sequence on $M \times (Y_{\gamma_\delta}/\mathbf{S}_\delta)$,

$$(*) 0 \mapsto \mathcal{I}_{Z_{\gamma_\delta}} \otimes \pi^*_\gamma L \mapsto \mathcal{O}_{M\times(Y_{\gamma_\delta}/\mathbf{S}_\delta)} \otimes \pi^*_\gamma L \mapsto \mathcal{O}_{Z_{\gamma_\delta}} \otimes p^*_{\gamma_\delta} L \mapsto 0.$$

Over here $\pi_\gamma : M \times (Y_{\gamma_\delta}/\mathbf{S}_\delta) \mapsto M$ and $p_{\gamma_\delta} : Z_{\gamma_\delta} \mapsto M$ are the natural projection maps.

[108] Consult subsection 6.4.

[109] Recall $f_{\delta,1} : M_{\delta+1} \mapsto M_1 = M$ is the composition of f_δ, $f_{\delta-1}$, $\ldots f_1$, where $f_i : M_{i+1} \mapsto M_i$ are the projection maps of the universal spaces, introduced in section 2.

On the other hand [110] $q_{3\delta}|_{Z_{3\delta}} : Z_{3\delta} \mapsto M^{[3\delta]}$ and the pre-image of the subset $\Pi_\delta(Y_{\gamma_\delta}/\mathbf{S}_\delta) \subset M^{[3\delta]}$ under $(q_{3\delta}|_{Z_{3\delta}})^{-1}$ (inside the universal sub-scheme $Z_{3\delta}(M)$) splits into a disjoint union of the form $\coprod_{i\leq\delta} Z_i$, where each Z_i represents a non-reduced sub-scheme of relative length 3 over the base $\Pi_\delta(Y_{\Gamma_\delta}/\mathbf{S}_\delta)$.

And there is a corresponding short exact sequence,

$$(**) 0 \mapsto \otimes_{1\leq i\leq\delta}\mathcal{I}_{Z_i}\otimes p_{3\delta}^*L \mapsto \mathcal{O}_{Z_{3\delta}(M)\cap q_{3\delta}^{-1}(\Pi_\delta(Y_{\Gamma_\delta}/\mathbf{S}_\delta))}\otimes p_{3\delta}^*L \mapsto \mathcal{O}_{\coprod_{1\leq i\leq\delta} Z_i}\otimes p_{3\delta}^*L \mapsto 0.$$

We claim that the push-forward the former short exact sequence $(*)$ of Z_{γ_δ} along $M\times(Y_{\gamma_\delta}/\mathbf{S}_\delta)\mapsto(Y_{\gamma_\delta}/\mathbf{S}_\delta)$ is isomorphic to the Π_δ^* pull-back of the $q_{3\delta *}$-push-forward of the short exact sequence $(**)$ on $\coprod_{1\leq i\leq\delta} Z_i$, due to the following commutative diagram of maps,

$$\begin{array}{ccc} Z_{\gamma_\delta} & \xrightarrow{q_{\gamma_\delta}} & Y_{\gamma_\delta}/\mathbf{S}_\delta \\ \Big\downarrow \breve{\pi}_\delta & & \Big\downarrow \Pi_\delta \\ Z_{3\delta}(M) & \xrightarrow{q_{3\delta}} & M^{[3\delta]} \end{array}$$

, with $\breve{\pi}_\delta : Z_{\gamma_\delta}\mapsto Z_{3\delta}(M)$ being the canonical inclusion.

Therefore we can identify the descend bundle $\mathbf{W}_{descend}\cong q_{\gamma_\delta *}p_{\gamma_\delta}^*L\otimes\mathbf{H}$, with $\Pi_\delta^* q_{3\delta *}p_{3\delta}^*(L)\otimes\mathbf{H}$. Moreover, because of the following commutative diagram on the bundle maps,

$$\begin{array}{ccc} \mathbf{H}^* & \longrightarrow & \pi^*_{|L|\times Y_{\gamma_\delta}/\mathbf{S}_\delta\mapsto Y_{\gamma_\delta}/\mathbf{S}_\delta} q_{\gamma_\delta *}p_{\gamma_\delta}^*L \\ \Big\downarrow & & \Big\downarrow \\ \mathbf{H}^* & \longrightarrow & \pi^*_{|L|\times M^{[3\delta]}\mapsto M^{[3\delta]}} q_{3\delta *}p_{3\delta}^*(L) \end{array}$$

the descend of the canonical section $s_{canon}|_{|L|\times Y_{\gamma_\delta}}$ corresponds to the Π_δ^*-pull-back of a section of $q_{3\delta *}p_{3\delta}^*(L)\otimes\mathbf{H}$ and a ray $\mathbf{l}$ in the projective space $|L|$ represents an algebraic curve in M singular along the sub-scheme $\coprod_{i\leq\delta} Spec(\mathcal{O}_{x_i,M}/m_{x_i}^2)$ iff the values of the canonical section s_{canon} at each of the points $\mathbf{l}\times\sigma(x_1\times x_2\times\cdots\times x_\delta)\in\mathbf{P}(\mathbf{V}_{canon})$, $\sigma\in\mathbf{S}_\delta$, vanishes.

By our assumption in the proposition, $\mathcal{M}_V\times_{M_\delta} Y_{\gamma_\delta}$ has been assumed to be a finite sub-scheme of $V\times M_\delta\subset X\times_{T(M)}\{t_L\}$. Because $\mathcal{M}_V\times_{M_\delta} Y_{\gamma_\delta}$ can be identified with the zero locus of s_{canon} in $V\times Y_{\gamma_\delta}$, then according to Section 14.1 of [F], one may define a localized top Chern class of $\pi_X^*\mathbf{W}_{canon}\otimes\mathbf{H}|_{V\times Y_{\gamma_\delta}}$ with respect to $s_{canon}|_{V\times Y_{\gamma_\delta}}$ inside $\mathcal{A}_0(\mathcal{M}_V\times_{M_\delta} Y_{\gamma_\delta})$. Because we have identified the descend bundle $\mathbf{W}_{descend}$ of $\pi_X^*\mathbf{W}_{canon}\otimes\mathbf{H}|_{X\times_{M_\delta} Y_{\gamma_\delta}}$ with $\Pi_\delta^* q_{3\delta *}p_{3\delta}^*(L)\otimes\mathbf{H}$, the image of $\mathcal{M}_V\times_{M_\delta} Y_{\gamma_\delta}$ in $(X\times_{M_\delta} Y_{\gamma_\delta})/\mathbf{S}_\delta$ can be identified with a finite sub-scheme $\subset |L|\times M_{2,0}^\delta$, denoted as W in the proof of proposition 20.

Since the localized top Chern class is defined by the local datum, i.e. the total Segre class of the normal cone of the zero locus and the restriction of the total Chern class of the vector bundle to the zero locus, the localized top Chern

[110]For the definitions of the maps $q_{3\delta}$, $p_{3\delta}$ and the Hilbert scheme $M^{[3\delta]}$, please consult the beginning of subsection 6.4.

class of $\mathbf{W}_{descend}$ along $\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta}/\mathbf{S}_\delta$ is equal to the localized top Chern class of $q_{3\delta *}p_{3\delta}^*(L) \otimes \mathbf{H}$ over W.

Since the quotient map $X \times_{M_\delta} Y_{\gamma_\delta} \mapsto X \times_{M_\delta} Y_{\gamma_\delta}/\mathbf{S}_\delta$ is an un-ramified covering map of degree $\delta!$, by proposition 14.1. (d).(iii). of [F], the degree of the localized top Chern class of $\pi_X^*\mathbf{W}_{canon} \otimes \mathbf{H}$ along the zero locus $\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta}$ is $\delta!$ times the localized top Chern class of $\mathbf{W}_{descend}$ along the quotient of zero locus $\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta}/\mathbf{S}_\delta$.

Because $W \mapsto pt$ factors through $W \subset V \times M_{2,0}^\delta \mapsto pt$, the degrees of the localized Chern class of $\mathbf{W}_{descend}|_{V \times Y_{\gamma_\delta}}$ along $\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta}/\mathbf{S}_\delta$ and the localized Chern class of $q_{3\delta *}p_{3\delta}^*(L) \otimes \mathbf{H}$ along W are equal and their common value is equal to

$$\int_{|V| \times M_{2,0}^\delta} c_{3\delta}(\mathbf{H} \otimes L_{3\delta}) = \int_{|V| \times M_{2,0}^\delta} c_1(\mathbf{H})^\delta \cap c_{2\delta}(L_{3\delta}) = \int_{M_{2,0}^\delta} c_{2\delta}(L_{3\delta}) = d_\delta(L).$$

Thus the degree of $[\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta}]$ is $\delta! d_\delta(L)$. The proof of proposition 19 is finished. □

6.4 The Finiteness Result of $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{M_n} Y_{\gamma_\delta}$

In this subsection, we survey the finiteness result based on Göttsche's argument in [Got].

Recall the following definition of k-very ampleness of a line bundle on M, due to [BS].

Definition 19 *A line bundle L on an algebraic manifold is k-very ample if for all length $k+1$ sub-scheme $Z \subset M$, the following restriction map $H^0(M, L) \mapsto H^0(M, \mathcal{O}_Z \otimes L)$ is surjective.*

The $1-$very ampleness is equivalent to the usual very ample condition, by page 120, prop. 7.3. on page 152 and remark 7.8.2. on page 158 of [Ha].

Definition 20 *Let $M_2^\delta \subset M^{[3\delta]}$ be the closure (with the reduced induced structure) of the locally closed subset $M_{2,0}^\delta$ which parametrizes sub-schemes of the form $\coprod_{i=1}^{\delta} Spec(\mathcal{O}_{M,x_i}/m_{M,x_i}^2)$, where $x_1, x_2, \cdots, x_\delta$ are distinct points in M.*

The symbol $M^{[n]}$ denote the Hilbert scheme of finite sub-schemes of length n on M, and let $Z_n \subset M \times M^{[n]}$ denote the universal family of sub-schemes with projection $p_n : Z_n(M) \mapsto M$ and $q_n : Z_n(M) \mapsto M^{[n]}$. Then $L_n = (q_n)_*(p_n)^*L$ is locally free of rank n on $M^{[n]}$. It is easy to see that M_2^δ is birational to $M^{[\delta]}$ and we set $d_\delta(L) = \int_{M_2^\delta} c_{2\delta}(L_{3\delta})$.

Recall the following proposition due to Göttsche. It is a word by word duplication of proposition 5.3. of [Got]. We include it here for the convenience of the readers.

Proposition 20 *(**Göttsche**) Assume L is $3\delta - 1$ very ample, then a general δ dimensional linear subsystem $V \subset |L|$ contains only finitely many curves $C_1, C_2, C_3, \cdots, C_s$ with $\geq \delta$ singularities. There exists positive integers $n_1, n_2, \cdots, n_s$ such that $\sum_i n_i = d_\delta(L)$. If furthermore L is $(5\delta-1)-$very ample (5-very ample if $\delta = 1$), then the C_i have precisely δ nodes as singularities.*

For completeness, we include its original proof here.
Proof (due to **Göttsche**): Assume that L is $(3\delta - 1)$-very ample. We apply the Thom-Porteous formula to the restrictions of the evaluation map $H^0(M, L) \otimes \mathcal{O}_{M^{[3\delta]}} \mapsto L_{3\delta}$ to M_2^δ and to $M_2^\delta - M_{2,0}^\delta$. As L is $(3\delta - 1)$-very ample, the evaluation map is surjective. Then ([F] ex. 14.3.2) applied to M_2^δ gives that for a general $\delta-$dimensional sub-linear system $V \subset |L|$ the class $d_n(L)$ is represented by the class of the finite scheme W of $Z \in M_2^\delta$ with $Z \subset D$ for $D \in V$. The scheme structure of W might be non-reduced. The application of ([F] ex. 14.3.2) to $M_2^\delta \backslash M_{2,0}^\delta$ and a dimension count give that W lies entirely in $M_{2,0}^\delta$.

Now assume that L is $(5\delta - 1)-$very ample. Let $V \subset |L|$ again be general $\delta-$dimensional subsystem of $|L|$. The Porteous formula applied to the restriction of $L_{3\delta+3}$ to $M_2^{\delta+1}$ and a dimension count shows that there will be no curves in V with more than δ singularities.

Let $M_{3,0}^\delta \subset M^{[5\delta]}$ be the locus of schemes of the form $Z_1 \cup Z_2 \cup Z_3 \cdots \cup Z_s$, where each Z_i is of the form $Spec(\mathcal{O}_{M,x_i}/(m^3 + xy))$ with x, y local parameters at x_i and let M_3^δ be the closure. If a curve C with precisely δ singularities does not contain a sub-scheme corresponding to a point in M_3^δ
$M_{3,0}^\delta$, then it has δ nodes as only singularities. It is easy to see that $M_{3,0}^\delta$ is smooth of dimension 4δ. Applying the Porteous formula to the restriction of $L_{5\delta}$ to M_3^δ
$M_{3,0}^\delta$ and a dimension count we see that all the curves in V with δ singularities have precisely δ nodes. □

In the following, we generalize Göttsche's argument to our context. Let $\mathbf{M}(E)E = \sum_{i \leq \delta} 2E_i$. Namely, $m_i = 2$ for all integers i, $1 \leq i \leq \delta$. A line bundle $L \mapsto M$ with $c_1(L) = C$ determines a unique point in the connected component of Picard group, $T(M)$, denoted as t_L. The fiber product $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{T(M)} \{t_L\}$ is the algebraic family moduli sub-space of curves in the fibers of the family $M_{\delta+1} \mapsto M_\delta$ projecting onto curves in $|L|$. Then there exists a natural map $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{T(M)} \{t_L\} \mapsto |L|$.

Proposition 21 *Let L be a $3\delta-1$-very ample line bundle over M. Let $\mathbf{M}(E)E = \sum_{1 \leq i \leq \delta} 2E_i$, and let $\mathcal{M}_V$ denote the pre-image of a general δ dimensional linear subsystem $V \subset |L|$ under $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{T(M)} \{t_L\} \mapsto |L|$. Then the fiber product $\mathcal{M}_V \times_{M_\delta} (M_\delta - \cup_{\Gamma \in \Delta(\delta) - \{\gamma_\delta\}} S_\Gamma)$ of $\mathcal{M}_V \mapsto M_\delta$ and $M_\delta - \cup_{\Gamma \in \Delta(\delta) - \{\gamma_\delta\}} S_\Gamma \subset M_\delta$ is a finite scheme and its image under the projection map to M_δ lies in the generic stratum Y_{γ_δ}.*

Proof: It is not hard to see that the image $\mathcal{M}_V \times_{M_n} Y_{\gamma_\delta} \mapsto V$ corresponds to all the curves in the linear sub system V which has at least δ distinct singularities. The space Y_{γ_δ} parametrizes all the ordered distinct δ points on M, and the

image of $\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta}$ in Y_{γ_δ} are the ordered δ-tuples of singular points of the curves. According to proposition 20 (by Gŏttsche), there are at most a finite number of singular curves in V with exactly δ distinct singularities.

This implies $\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta}$ to be a finite scheme. On the other hand, the image of $\mathcal{M}_V \mapsto M_\delta$ may have non-trivial intersections with the various subsets $S_\Gamma, \Gamma \in \Delta(\delta) - \{\gamma_\delta\}$.

To prove the proposition, it suffices to show that the image $\mathcal{M}_V \mapsto M_\delta$ intersect with Y_Γ trivially for all the chain-like [111] $\Gamma \in adm_2(\delta)$. Then $\mathcal{M}_V \times_{M_\delta} (M_\delta - \cup_{\Gamma \in \Delta(\delta) - \{\gamma_\delta\}} S_\Gamma)$ can be identified with the space $\mathcal{M}_V \times_{M_\delta} Y_{\gamma_\delta}$ and it has been shown to be a finite scheme which projects into Y_{γ_δ} automatically.

To show that the image of $\mathcal{M}_V \mapsto M_\delta$ avoids all the Y_Γ for chain-like Γ, we fix an arbitrary $\bar{b} \in Y_\Gamma$ and show that $\bar{b}$ is not included in the image of $\mathcal{M}_V \mapsto M_\delta$ for generic choices of V.

Lemma 27 *Let $\Gamma \in adm_2(\delta)$ be a chain-like admissible graph. The fiber above the point $\bar{b} \in Y(\Gamma) \subset M_\delta$ of $M_{\delta+1} \mapsto M_\delta$ determines a $\delta-$consecutive blowing ups of M, denoted by $\tilde{M}$. As usual, let $E_1, E_2, \cdots, E_\delta$ denote the δ exceptional divisors in $\tilde{M}$ of the blowing down map $f : \tilde{M} \mapsto M$. Then $f_*\mathcal{O}_{\tilde{M}}(-2\sum_{i\leq\delta} E_i)$ is an ideal sheaf $\subset \mathcal{O}_M$ of a finite sub-scheme of M of length 3δ.*

Proof of Lemma 27: Firstly we prove that $f_*\mathcal{O}_{\tilde{M}}(-2\sum_{i\leq\delta} E_i)$ is an ideal sheaf by showing that it is a sub-sheaf of $\mathcal{O}_M$.

To see this, we notice that $\mathcal{O}_{\tilde{M}}(-2\sum_{i\leq\delta} E_i) \subset \mathcal{O}_{\tilde{M}}$ and thus $f_*\mathcal{O}_{\tilde{M}}(-2\sum_{i\leq\delta} E_i) \subset f_*\mathcal{O}_{\tilde{M}}$. On the other hand, the exceptional divisors of $f : \tilde{M} \mapsto M$ are all rational, this implies that $f_*\mathcal{O}_{\tilde{M}} = \mathcal{O}_M$. Thus, $f_*\mathcal{O}_{\tilde{M}}(-2\sum_{i\leq\delta} E_i)$ is a sub-sheaf of $\mathcal{O}_M$. Let Z be [112] the sub-scheme of M defined by the ideal sheaf $\mathcal{I}_Z = f_*\mathcal{O}_{\tilde{M}}(-2\sum_{i\leq\delta} E_i)$.

Secondly, we prove that the length of Z is bounded by 3δ from above. This is achieved by an induction argument on δ. For $\delta = 1$, there is a unique exceptional divisor E_1. Let $x \in M$ be the blowing up point. It is easy to see that $\mathcal{O}_Z \cong \mathcal{O}_M/\mathcal{I}_Z \cong \mathcal{O}_x/m_x^2$ and Z is of length $3 = 3 \cdot 1 = 3 \cdot \delta$.

For $\delta > 1$, suppose that for all the smooth algebraic surfaces M and all δ-consecutive blowing ups, $\tilde{M}$, of M, the ideal sheaf $f_*\mathcal{O}_{\tilde{M}}(-2\sum_{i\leq\delta} E_i)$ is known to define a length $\leq 3\delta$ sub-scheme of M, we would like to show that for $\delta+1$, and the $\delta+1$-consecutive blowing ups $\check{M}$ of M, the ideal sheaf $\check{f}_*\mathcal{O}_{\check{M}}(-2\sum_{i\leq\delta+1} E_i)$ defines a sub-scheme of M of length $\leq 3(\delta+1)$.

We notice that $\check{f} : \check{M} \mapsto M$ can be factored into $\check{\bar{f}} : \check{M} \mapsto \bar{M}$ and $\bar{f} : \bar{M} \mapsto M$, where $\bar{M}$ is a one-point blowing up of M, with the exceptional divisor E_1, and $\check{M}$ can be constructed from $\bar{M}$ by δ-consecutive blowing ups.

By induction hypothesis, $\check{\bar{f}}_*\mathcal{O}_{\check{M}}(-2\sum_{2\leq i\leq\delta+1} E_i)$ defines an ideal sheaf on $\bar{M}$ of the sub-scheme, $\bar{Z} \subset \bar{M}$ of length $\leq 3\delta$.

Then

$$\check{\bar{f}}_*\mathcal{O}_{\check{M}}(-2\sum_{1\leq i\leq\delta+1} E_i) = \check{\bar{f}}_*\big(\mathcal{O}_{\check{M}}(-2\sum_{2\leq i\leq\delta+1} E_i)\otimes\check{\bar{f}}^*\mathcal{O}_{\check{M}}(-2E_1)\big) = \mathcal{I}_{\bar{Z}}\otimes\mathcal{O}_{\bar{M}}(-2E_1),$$

[111] Consult definition 6 and the comment afterward.

[112] The Z has nothing to do with the various Z used in the previous sections.

and

$$\check{f}_*\mathcal{O}_{\check{M}}(-2\sum_{1\leq i\leq\delta+1}E_i) = \bar{f}_*\big(\check{\bar{f}}_*\mathcal{O}_{\check{M}}(-2\sum_{1\leq i\leq\delta+1}E_i)\big) = \bar{f}_*\big(\mathcal{I}_{\bar{Z}}\otimes\mathcal{O}_{\bar{M}}(-2E_1)\big).$$

By using the exactness of the sequence, $0\mapsto\bar{f}_*\mathcal{I}_{\bar{Z}}\mapsto\bar{f}_*\mathcal{O}_{\bar{M}}\mapsto\bar{f}_*\mathcal{O}_{\bar{Z}}$,

and the fact $\bar{f}_*\mathcal{O}_{\bar{M}}(-2E_1)$ being an ideal sheaf of M of co-length 3, the length of the sub-scheme defined by $\bar{f}_*\big(\mathcal{I}_{\bar{Z}}\otimes\mathcal{O}_{\bar{M}}(-2E_1)\big)$ is bounded by $length(\bar{Z})+3 = 3\delta+3 = 3(\delta+1)$ from above.

Thirdly, we show that the equality is saturated, namely $length(Z) = 3\delta$ for all $\delta\in\mathbf{N}$, when $\Gamma\in\Delta(\delta)$ is a chain-like admissible graph $\in adm_2(\delta)$. We prove this by contradiction. We know that for $\delta=1$ the equality always saturates. Suppose that there is some chain-like admissible graph Γ and for some $\bar{b}\in Y_\Gamma$, the fiber $\tilde{M}$ of $M_{\delta+1}\mapsto M_\delta$ above $\bar{b}$, $f:\tilde{M}\mapsto M$, such that the ideal sheaf $f_*\mathcal{O}_{\tilde{M}}(-2\sum_{i\leq\delta}E_i) = \mathcal{I}_Z$ is of length $<3\delta$. We may assume additionally that the smallest natural number $\delta_0>1$ satisfying the above condition has been chosen. I.e. for all natural numbers δ smaller than δ_0 and any chain-like admissible graphs $\in adm_2(\delta)$, the ideal sheaf always defines length 3δ sub-schemes in M.

Apparently the question is of local nature, so we may assume that we are blowing up consecutively at the origin $\mathbf{0}\in\mathbf{C}^2=M$. Suppose that the connected graph $\Gamma\in adm_2(\delta_0)$ is a linear chain and the $i-$th vertex is the unique direct descendent of the $i-1-$th vertex, for all $2\leq i\leq\delta_0$. Let x,y be the affine coordinates around the origin $\mathbf{0}$.

Consider the blowing down of $\tilde{M}$ along the last exceptional divisor E_{δ_0}, $\acute{f}:\tilde{M}\mapsto\acute{M}$. Then $\acute{M}$ is a δ_0-1 consecutive blowing ups of M at $\mathbf{0}$. Define $\mathcal{I}_{Z_0}=\mathcal{O}_{\acute{M}}$, $\mathcal{I}_{Z_1}=\acute{f}_*\mathcal{O}_{\tilde{M}}(-E_{\delta_0})$ and $\mathcal{I}_{Z_3}=\acute{f}_*\mathcal{O}_{\tilde{M}}(-2E_{\delta_0})$. Let $\mathcal{I}_{Z_2}$ be an ideal sheaf of co-length 2 in-between $\mathcal{I}_{Z_1}$ and $\mathcal{I}_{Z_3}$, i.e. $\mathcal{I}_{Z_1}\supset\mathcal{I}_{Z_2}\supset\mathcal{I}_{Z_3}$. Then for $\check{f}:\acute{M}\mapsto M$, we have

$$f_*\mathcal{O}_{\tilde{M}}(-2\sum_{i\leq\delta_0-1}E_i-E_{\delta_0})\cong\check{f}_*\big(\mathcal{O}_{\acute{M}}(-2\sum_{i\leq\delta_0-1}E_i)\otimes\mathcal{I}_{Z_1}\big)\supset\check{f}_*\big(\mathcal{O}_{\acute{M}}(-2\sum_{i\leq\delta_0-1}E_i)\otimes\mathcal{I}_{Z_2}\big)$$

$$\supset\check{f}_*\big(\mathcal{O}_{\acute{M}}(-2\sum_{i\leq\delta_0-1}E_i)\otimes\mathcal{I}_{Z_3}\big)\cong\check{f}_*\mathcal{O}_{\tilde{M}}(-2\sum_{i\leq\delta_0}E_i).$$

The minimality of δ_0 implies that $f_*\mathcal{O}_{\tilde{M}}(-2\sum_{i\leq\delta_0-1}E_i)$ is of co-length $3(\delta_0-1)$ in $\mathcal{O}_M$. Because $f_*\mathcal{O}_{\tilde{M}}(-2\sum_{i\leq\delta_0-1}E_i)\supset f_*\mathcal{O}_{\tilde{M}}(-2\sum_{i\leq\delta_0}E_i)$ is of co-length <3, there must be some $a\in\{0,1,2\}$ such that $\check{f}_*\big(\mathcal{O}_{\acute{M}}(-2\sum_{i\leq\delta_0-1}E_i)\otimes\mathcal{I}_{Z_a}\big)=\check{f}_*\big(\mathcal{O}_{\acute{M}}(-2\sum_{i\leq\delta_0-1}E_i)\otimes\mathcal{I}_{Z_{a+1}}\big)$.

Define $\check{f}_*\big(\mathcal{O}_{\acute{M}}(-2\sum_{i\leq\delta_0-1}E_i)\otimes\mathcal{I}_{Z_a}\big)=\mathcal{I}_Z$ for such an a. Consider a polynomial $g(x,y)\in\mathbf{C}[x,y]$ vanishing along the sub-scheme Z.

The sheaf identification $\check{f}_*\big(\mathcal{O}_{\acute{M}}(-2\sum_{i\leq\delta_0-1}E_i)\otimes\mathcal{I}_{Z_a}\big)=\mathcal{I}_Z$ induces an identification $\psi_a:\Gamma(M,\mathcal{I}_Z)\xrightarrow{\cong}\Gamma(\acute{M},\check{f}_*\big(\mathcal{O}_{\acute{M}}(-2\sum_{i\leq\delta_0-1}E_i)\otimes\mathcal{I}_{Z_a}\big)$.

Then the image $\psi_a(g)$ of g in $\Gamma(\acute{M}, \breve{f}_*(\mathcal{O}_{\acute{M}}(-2\sum_{i\leq\delta_0-1} E_i) \otimes \mathcal{I}_{Z_a})$ defines a zero locus in $\acute{M}$. Suppose that g can be chosen such that $\psi_a(g)$ does not vanish identically on $E_1, E_2, \cdots, E_{\delta_0-1}$, then the zero locus $Z(\psi_a(g)) = \{x|\psi_a(g)(x) = 0, x \in \acute{M}\}$ in $\acute{M}$ is nothing but the strict transform of $Z(g) = \{x|g(x) = 0, x \in M\}$ under the $\delta_0 - 1-$consecutive blowing ups.

By the choice of a we know that $\breve{f}_*\mathcal{O}_{\acute{M}}(-2\sum_{i\leq\delta_0-1} E_i) \otimes \mathcal{I}_{Z_{a+1}} = \mathcal{I}_Z$ as well. So the defining equation $\psi_a(g)$ vanishes along the length $a+1$ sub-scheme $Z_{a+1} \supset Z_a$ automatically.

We demonstrate the existence of some counter-example violating the above assertion in the following lemma 28. After it is achieved, then the co-length of $f_*\mathcal{O}_{\tilde{M}}(-2\sum_{1\leq i\leq\delta_0} E_i)$ has to be exactly $3\delta_0$ and therefore the minimal δ_0 violating the saturation condition can never exist. Then the proof of lemma 27 is finished. $\Box$

The following lemma supports the counter-example needed in the proof of lemma 27.

Lemma 28 *Let $M = \mathbf{C}^2$ and let Γ be a connected chain-like admissible graph in $adm_2(\delta_0)$. As above fix a point $\bar{b} \in Y_\Gamma$ and therefore a δ_0-consecutive blowing up of M at its origin $\mathbf{0}$. Let a be chosen as above and let Z_a, Z_{a+1} be the length a and $a+1$ sub-schemes of $\acute{M}$ defined above. Given any nonzero $g \in H^0(M, \mathcal{I}_Z)$, let $\acute{g} \in H^0(\acute{M}, \mathcal{O}_{\acute{M}})$ be the defining equation of the strict transform of the locus $Z(g)$ in $\acute{M}$ (well-defined up to a $\mathbf{C}^*$ multiplication). Then there exists a nonzero $\mathbf{g} \in H^0(\mathbf{C}^2, \mathcal{I}_Z)$ such that $\acute{\mathbf{g}} \in H^0(\acute{M}, \mathcal{O}_{\acute{M}}(-2\sum_{1\leq i\leq\delta_0-1} E_i) \otimes \mathcal{I}_{Z_a})$ but $\acute{\mathbf{g}} \notin H^0(\acute{M}, \mathcal{O}_{\acute{M}}(-2\sum_{1\leq i\leq\delta_0-1} E_i) \otimes \mathcal{I}_{Z_{a+1}})$.*

Proof of lemma 28: By the embedded resolutions of singular curves in algebraic surfaces, (for example consult 8B, page 160-166 of [Mum]), the rational double point in a singular curve in $\mathbf{C}^2$ defined by the equation $x^2 = y^{2(\delta_0-1)}$ can be resolved into smooth points by consecutively blowing up $\delta_0 - 1$ times, each upon the unique singular point of the intermediate strict transforms.

Algebraically blowing up a point corresponds to replacing the coordinates (x, y) by $(x', y')(= \frac{x}{y}, y)$ (or $(x', y') = (x, \frac{y}{x})$) in the defining equations. And under the first set of change of variables the equation becomes

$$x^2 - y^{2(\delta_0-1)} = (\frac{x}{y}\cdot y)^2 - y^{2(\delta_0-1)} = y'^2((\frac{x}{y})^2 - y'^{2(\delta_0-2)}) = y'^2(x'^2 - y'^{2(\delta_0-2)}).$$

Firstly, the strict transformation of the zero locus in the one-point blowing up of M at the origin, defined by $x'^2 = y'^{2(\delta_0-2)}$, has a rational double point ($A_{2\delta_0-5}$ singularity) at $x' = y' = 0$ and it intersects with the exceptional divisor (defined locally by $y' = 0$ here) with a singular multiplicity $\mu = 2$. By induction one realizes that the original singular curve gets resolved into a smooth curve after $\delta_0 - 1$ consecutively blowing ups and the resolved smooth curve intersects with the last exceptional $\mathbf{CP}^1$ (dual to E_{δ_0-1}) at two distinct points. This can be seen by observing that the $\delta_0 = 2$ case corresponds to nothing but the

ordinary double (nodal singular) point. The blowup sequence determined by $\bar{b} \in Y_\Gamma$ determines a sequence of $\delta_0 - 1$ points in a sequence of the first $\delta_0 - 1$ exceptional $\mathbf{P}^1$, each representing an exceptional divisor in the intermediate blowing ups. Apparently the blowing-up centers in resolving $x^2 = y^{2(\delta_0-1)}$ to a smooth curve may not be identical to the first $\delta_0 - 1$ blowing-up centers determined by $\bar{b} \in Y_\Gamma$. On the other hand, the change of variables on page 161, in the subsection (8.6) of [Mum] allows us to move the locations of the intermediate singularities that are blown up. We proceed as the following.

Firstly notice that it requires at least $2\delta_0 - 1$ affine coordinate systems to cover the $\delta_0 - 1$ distinct exceptional $\mathbf{P}^1$ of $\acute{M} \mapsto M$ and the punctured neighborhood of M around $\mathbf{0}$. Let $(\mathbf{x}, \mathbf{y})$ be the affine coordinate of M at $\mathbf{0}$ and let (x_{2i-1}, y_{2i-1}), (x_{2i}, y_{2i}), $1 \leq i \leq \delta_0 - 1$ be the dual affine coordinates on the neighborhoods of the $i-$th exceptional $\mathbf{P}^1$. For a fixed i, they satisfy the following transition rules $x_{2i-1}y_{2i-1} = x_{2i}$, $\frac{1}{x_{2i-1}} = y_{2i}$. The locus y_{2i-1} or $x_{2i} = 0$ corresponds to the $i-$th $\mathbf{P}^1$. To determine the transitions of coordinates among different i, it suffices to work out the transition for the adjacent pairs.

The blowing up sequence determined by $\bar{b}$ determines δ_0-1 points $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{\delta_0-1}$ in the $\delta_0 - 1$ exceptional $\mathbf{P}^1$ of $\acute{M} \mapsto M$, respectively.

Firstly focus on the exceptional $\mathbf{P}^1$ representing the $\delta_0 - 1$-th divisor E_{δ_0-1}. Because $Z_1 = \mathbf{v}_{\delta_0-1}$ is a point in this $\mathbf{P}^1$, either it is covered by the coordinate system $(x_{2\delta_0-3}, y_{2\delta_0-3})$ with $y_{2\delta_0-3} = 0$, or it is at the origin of $(x_{2\delta_0-2}, y_{2\delta_0-2})$ coordinate system.

By choosing either $(u, v) = (x_{2\delta_0-3}, y_{2\delta_0-3})$ or $(u, v) = (y_{2\delta_0-2}, x_{2\delta_0-2})$, we assume that (u, v) is a coordinate system around the $\delta_0 - 1-$th exceptional $\mathbf{P}^1$ containing th point $\mathbf{v}_{\delta_0-1}$ such that $v = 0$ defines the exceptional curve locally and u is a local uniformizer of the exceptional curve. We choose the constants $\alpha, \beta \in \mathbf{C}$, and $A \in \mathbf{C}$ in the quadratic polynomial $g_{\delta_0-1}(u, v) = (u-\alpha)(u-\beta) + Av$ according to the value of $a \in \{0, 1, 2\}$.

Notice that when $\alpha \neq \beta$, the equation $v^2 g_{\delta_0-1}(\frac{u}{v}, v) = u^2 - (\alpha+\beta)uv + \alpha\beta v^2 + Av^3 = 0$ represents a curve with a rational double point at the origin of (u, v). We know that $u = \alpha$, $u = \beta$ are the affine coordinates of the two intersection points of the resolved smooth curve with E_{δ_0-1}, locally defined by $v = 0$.

Case 0: Suppose that $a = 0$, then set $A = 0$ and choose generic α and β to move the two intersection points of the resolved smooth curve with the exceptional curve E_{δ_0-1} away from the blown up point $Z_1 = \mathbf{v}_{\delta_0-1}$ of$\acute{f} : \tilde{M} \mapsto \acute{M}$.

Case 1: Suppose that $a = 1$. Firstly choose $\beta = \beta_0$ such that $u = \beta_0$ is the affine coordinate of blown up point Z_1 in $\acute{M}$. Thus the smooth curve resolved from the nodal curve locally defined by $v^2 g_{\delta_0-1}(\frac{u}{v}, v) = u^2 - (\alpha+\beta)uv + \alpha\beta v^2 + Av^3 = 0$ vanishes along the length one sub-scheme Z_1. Notice that in terms of the local uniformizers $(u-\beta_0), v$ at $(\beta_0, 0)$ the first jets of $g_{\delta_0-1}(u, v)$ are determined by $\alpha - \beta_0$ and A. We choose a generic α, $\alpha \neq \beta_0$ and A such that the resolved smooth curve does not vanish along the sub-scheme Z_2. This is possible because the length 2 sub-scheme $Z_2 \subset \acute{M}$ determines a tangent direction of $\acute{M}$ at Z_1 and the generic choices of α and A can prevent the locus $g_{\delta_0-1}(u, v) = 0$ from

being tangent to this tangent direction specified by Z_2 at Z_1.

Case 2: Suppose that $a = 2$. As before we choose $\beta = \beta_0$ such that $u = \beta_0$ is the affine coordinate of the blown up point $Z_1 \subset \mathbf{P}^1$. The rest of the discussion depends on Z_2 explicitly. If the length two sub-scheme Z_2 represents the tangent direction to E_{δ_0-1} at Z_1, then we take $\alpha = \beta_0$ and $A \neq 0$. If Z_2 determines a tangent direction of $\acute{M}$ at Z_1 other than the tangent direction of E_{δ_0-1} at Z_1, then we choose a pair of non-identically zero $\alpha \neq \beta_0$ and A such that the conic determined by the equation $g_{\delta_0-1}(u,v) = 0$ is tangent to this given direction specified by Z_2. Because the first jets of g_{δ_0-1} at $(u,v) = (\beta_0, 0)$ are not identically zero, it is apparent that the polynomial g_{δ_0-1} does not vanish along Z_3.

Definition 21 *A polynomial $f(x,y) \in \mathbf{C}[x,y]$ is said to be leaded by the variable x of degree two if it only contains monomials $x^s y^t$ with $0 \leq s \leq 2$. It is said to be leaded by the variable y of degree two if it only contains monomials of the type $x^s y^t$ with $0 \leq t \leq 2$.*

It is easy to observe that if a polynomial is leaded by x (or by y) of degree two, then $y^2 f(\frac{x}{y}, y)$ (or $x^2 f(x, \frac{y}{x})$), $f(x + a\cdot y, y)$ (or $f(x, y + bx)$) are still leaded by x (or by y) as well.

We employ the following inductive procedure with decreasing i, $1 \leq i \leq \delta_0 - 2$, to determine the transition maps between different coordinate charts and g_i. Starting from $g_{\delta_0-1} = g_{i+1}$ with $i = \delta_0 - 2$.

Case I: If g_{i+1} is leaded by x_{2i+1} or y_{2i+2}, then consider the following transition rule.

(i). Suppose that the point $\mathbf{v}_i$ is in the open subset of the $i-th$ $\mathbf{P}^1$ covered by the coordinate system $(x_{2i-1}, 0)$ with an affine coordinate $(\alpha_{2i-1}, 0)$, then set $(x_{2i-1} - \alpha_{2i-1}) = y_{2i+1}x_{2i+1}$, $y_{2i-1} = y_{2i+1}$; $(x_{2i-1} - \alpha_{2i-1}) = x_{2i+2}$, $y_{2i-1} = x_{2i+2}y_{2i+2}$ for the coordinate transitions.

(ii). Suppose that the point $\mathbf{v}_i$ is not in the open subset of $\mathbf{P}^1$ covered by the coordinate system $(x_{2i-1}, 0)$, then it must be covered by the affine coordinate system $(0, y_{2i})$ with an affine coordinate $y_{2i} = 0$. We set $y_{2i} = x_{2i+2}y_{2i+2}$, $x_{2i} = x_{2i+2}$; $y_{2i} = y_{2i+1}$, $x_{2i} = x_{2i+1}y_{2i+1}$ for the coordinate transitions.

It is apparent that our choices of transition maps are consistent with the transitions of dual coordinates $(x_{2i+1}, y_{2i+1}) \leftrightarrow (x_{2i+2}, y_{2i+2})$ defined earlier.

Define g_i as $(x_{2i-1} - \alpha_i)^2 g_{i+1}(\frac{x_{2i-1}-\alpha_i}{y_{2i-1}}, y_{2i-1})$ (in alternative (i)) or $g_i = y_{2i}^2 g_{i+1}(\frac{x_{2i}}{y_{2i}}, y_{2i})$ (in alternative (ii)) if g_{i+1} is leaded by x_{2i+1} of degree two.

Define g_i as $(x_{2i-1} - \alpha_i)^2 g_{i+1}(x_{2i-1} - \alpha_i, \frac{y_{2i-1}}{x_{2i-1}-\alpha_i})$ (in the alternative (i)) or $g_i = (x_{2i})^2 g_{i+1}(x_{2i}, \frac{y_{2i}}{x_{2i}})$ (in the alternative (ii)) if g_i is leaded by y_{2i+2} of degree two.

Case II:

If g_{i+1} is leaded by y_{2i+1} or x_{2i+2}, we consider the following alternative scheme instead.

(i)'. If $\mathbf{v}_i$ is in the open subset of the $i-$th $\mathbf{P}^1$ covered by the coordinate system $(0, y_{2i})$ with an affine coordinate $(0, \beta_{2i})$, then set $(y_{2i} - \beta_{2i}) = x_{2i+1}y_{2i+1}$, $x_{2i} = x_{2i+1}$; $(y_{2i} - \beta_{2i}) = y_{2i+2}$, $x_{2i} = x_{2i+2}y_{2i+2}$ for the coordinate transitions.
(ii)'. If the point $\mathbf{v}_i$ is not in the open subset of $\mathbf{P}^1$ covered by the coordinate system $(0, y_{2i})$, it must be covered by the affine coordinate system $(x_{2i-1}, 0)$ with an affine coordinate $x_{2i-1} = 0$. Then we set $y_{2i} = x_{2i+1}y_{2i+1}$, $x_{2i} = x_{2i+1}$; $y_{2i} = y_{2i+2}$, $x_{2i} = x_{2i+2}y_{2i+2}$ for the coordinate transitions.

Define g_i to be $(y_{2i} - \beta_{2i})^2 g_{i+1}(\frac{x_{2i}}{y_{2i}-\beta_{2i}}, y_{2i} - \beta_{2i})$ (in the alternative (i)') or $y_{2i}^2 g_{i+1}(\frac{x_{2i}}{y_{2i}}, y_{2i})$ (in alternative (ii)') if g_{i+1} is leaded by x_{2i+2}.

Define g_i to be $x_{2i}^2 g_{i+1}(x_{2i}, \frac{y_{2i}-\beta_{2i}}{x_{2i}})$ (in the alternative (i)') or $x_{2i}^2 g_{i+1}(x_{2i}, \frac{y_{2i}}{x_{2i}})$ (in alternative (ii)') if g_{i+1} is leaded by y_{2i+1}.

It is easy to check that the two-variable polynomial g_i is still leaded by one of its variables of degree two.

If $i \geq 1$, decrease i by one, $i \mapsto i - 1$, and repeat the above process until $i = 0$.

Finally when $i = 0$, jump out of the defining loops and define[113] $(\mathbf{x}, \mathbf{y}) = (x_1 y_1, y_1)$ and $(\mathbf{x}, \mathbf{y}) = (x_2, x_2 y_2)$. Define $g_0(\mathbf{x}, \mathbf{y}) = \mathbf{y}^2 g_1(\frac{\mathbf{x}}{\mathbf{y}}, \mathbf{y})$ if g_1 is leaded by x_1 or x_2 of degree two. Define $g_0(\mathbf{x}, \mathbf{y}) = \mathbf{x}^2 g_1(\mathbf{x}, \frac{\mathbf{y}}{\mathbf{x}})$ if g_1 is leaded by y_1 or y_2 of degree two.

The union of the zero loci in $\acute{M}$ defined by $g_i = 0$, $0 \leq i \leq \delta_0 - 1$ on the δ_0 different coordinate charts form an algebraic curve intersecting E_i, $1 \leq i \leq \delta_0 - 1$, with multiplicity two. By our construction of $g_{\delta_0 - 1}$ above, it vanishes along Z_a but not along Z_{a+1}. Because this curve is a divisor in $\acute{M}$, it is defined by a global section $\acute{g}_0 \in H^0(\acute{M}, \mathcal{O}_{\acute{M}}(-2\sum_{1\leq i\leq \delta_0-1} E_i) \otimes \mathcal{I}_{Z_a})$. By our construction we know that $\acute{g}_0 \notin H^0(\acute{M}, \mathcal{O}_{\acute{M}}(-2\sum_{1\leq i\leq \delta_0-1} E_i) \otimes \mathcal{I}_{Z_{a+1}})$.

The explicit form of $\acute{g}_0$ on all $2\delta_0 - 1$ coordinate charts can be determined by the $g_i, 0 \leq i \leq \delta_0 - 1$ (on the δ_0 charts) and the transition maps among dual charts covering E_i. On the other hand, $\acute{f}|_{\acute{M} - \cup_{1\leq i\leq \delta_0-1} E_i} : \acute{M} - \cup_{1\leq i\leq \delta_0-1} E_i \mapsto M - \mathbf{0}$ is an isomorphism under the blowing down map. Under this identification $(\mathbf{x}, \mathbf{y}) = \acute{f}^*(x, y)$, $g_0(x, y)$ defines a polynomial $\in H^0(M, \mathcal{O}_M) \cong \mathbf{C}[x, y]$. Because $\acute{g}_0$ is a global section of $\mathcal{O}_{\acute{M}}(-2\sum_{1\leq i\leq \delta_0-1} E_i) \otimes \mathcal{I}_{Z_a}$, $g_0 \in H^0(M, \acute{f}_*(\mathcal{O}_{\acute{M}}(-2\sum_{1\leq i\leq \delta_0-1} E_i) \otimes \mathcal{I}_{Z_a})) = H^0(M, \mathcal{I}_Z)$.

The pair $(\acute{g}_0, g_0)$ satisfy the requirement in lemma 28. So the proof of lemma 28 is finished. $\Box$

From lemma 27, we know that for all chain-like $\Gamma_0 \in adm_2(\delta)$, the scheme Z defined by the ideal sheaf $\mathcal{I}_Z = f_* \mathcal{O}_{\tilde{M}}(-2\sum_{i\leq \delta} E_i)$ is of length 3δ.

This implies that for any chain-like $\Gamma_0 \in adm_2(\delta)$, a point $\bar{b} \in Y_{\Gamma_0} \subset M_{\delta_0}$ is in the image of $\mathcal{M}_V \times_{M_\delta} (M_\delta - \cup_{\Gamma \in \Delta(\delta)} S_\Gamma)$ if there exists a point $c \in \mathcal{M}_V$ above $\bar{b}$ such that its image inside the δ dimensional linear sub-system V under $\mathcal{M}_V \mapsto V$ lies in the kernel $H^0(M, \mathcal{I}_Z \otimes L)$ of $H^0(M, L) \mapsto H^0(M, \mathcal{O}_Z \otimes L)$. I.e. the corresponding curve represented by the point c vanishes along the length 3δ sub-scheme Z.

[113] Because we blow up $\mathbf{0} \in M$.

By the $3\delta - 1$-very ampleness condition on L and the argument of Göttsche's proposition 20, for generic choices of $V \subset |L|$ there can be no such curve. So the proof of proposition 21 is complete. $\square$

7 Appendix: The Relationship with the Gromov-Witten Invariant

In the previous section, we have given an algebraic proof that the "number of δ-nodes nodal curves" in a general δ dimensional linear-subsystem of $|L|$ can be expressed as a universal polynomial of $L \cdot L, L \cdot c_1(\mathbf{K}_M), c_1(\mathbf{K}_M)^2, c_2(M)$. This "number of nodal curves" is understood in the sense of Göttsche (see proposition 20) and our proof involves the various constructions in the algebraic family Seiberg-Witten theory. The reader working on Gromov-Witten invariant may desire to understand the relationship between the "family Seiberg-Witten invariant count" and the Gromov-Witten invariant count. As sometimes it may lead to some misunderstanding of the result, so we offer some clarification here.

Firstly, for the difference between the usual "algebraic" Seiberg-Witten invariant (over $B = pt$) and the topological version of Seiberg-Witten invariant (which is equivalent to the "right genus" Gromov-Witten invariant of an algebraic surface by [T1], [T2], [T3] and [IP]), please consult sub-section 4.3.1 of [Liu3]. So we will focus on the difference of our "number of nodal curves" and the usual "wrong genera" Gromov-Witten invariant.

Given a holomorphic line bundle L on M, the adjunction formula, $C^2 + c_1(\mathbf{K}_M) \cdot C = 2g(C) - 2$ with $C = c_1(L)$, predicts a preferred genus of curves in $|L|$. An identical adjunction formula holds in the pseudo-holomorphic category as well. Taubes had used the pseudo-holomorphic curve counting of the preferred genus in developing his "SW=Gr" theorem, [T1], [T2], [T3] etc.

On the other hand, in the standard GW invariant the genus of the source curve is not pre-determined by the class $C = c_1(L) \in H^2(M, \mathbf{Z})$. In fact, for all $g \in \mathbf{N}$, it makes sense to define the genus g Gromov-Witten invariant $GW_g(C)$ which enumerates the virtual number of (pseudo-)holomorphic maps representing $C \in H^2(M, \mathbf{Z})$ from source curves with genus g.

The fundamental observation which relates the "nodal curve counting" with the number $GW_g(C)$ is that a genus $g < g(C)$ (pseudo)-holomorphic curve tends to develop $g(C) - C$ nodal singularities. It is because a (pseudo)-holomorphic map from a genus g, $g < g(C)$ curve, Σ_g mapping into M cannot be embedded (or it violates the adjunction formula) and tends to develop isolated singularities in its image (if it is not multiple-covered or bubbling off multiple coverings of two spheres). The nodal curve singularities are preferred because of dimension reason. Suppose that the image of Σ_g has developed singularities at $x_1, x_2, \cdots, x_k \in M$ with singular multiplicities $m_1, m_2, \cdots, m_k$ for some $k \in \mathbf{N}$.

Then the adjunction formula for singular curves, exercise 1.3 and corollary 3.7 of chapter V of [Ha], implies that

$$2g-2+\sum_{i\leq k} m_i(m_i-1) = C^2 + c_1(\mathbf{K}_M)\cdot C.$$

On the other hand for $g \geq 2$, the expected dimension of the Gromov-Witten moduli space is equal to

$$\int_{\Sigma_g} c_1(M) - 2(g-1) + dim_{\mathbf{C}}\mathcal{M}_g = -c_1(\mathbf{K}_M)\cdot C + \frac{C^2 + c_1(\mathbf{K}_M)\cdot C}{2} - \sum_{i\leq k}\frac{m_i(m_i-1)}{2} = \frac{C^2 - c_1(\mathbf{K}_M)\cdot C}{2} - \sum_{i\leq k}\frac{m_i(m_i-1)}{2},$$

where $\frac{C^2-c_1(\mathbf{K}_M)\cdot C}{2}$ is both (i). the expected dimension of Gromov-Taubes moduli space (see Taubes [T3]) and (ii). The C dependent term of the surface Riemann-Roch formula and closely related to the dimension of the (non-)linear system associated to a given $C \in H^2(M, \mathbf{Z})$. On the other hand, the expected dimension of algebraic curves carrying k singularities with multiplicities $m_1, m_2, \cdots, m_k$ is at most $\frac{C^2-c_1(\mathbf{K}_M)\cdot C}{2} - \sum_{1\leq i\leq \delta}(\frac{m_i^2+m_i}{2} - 2)$. Because $m(m-1) \leq m(m+1)-4$ for $m \geq 2$ and the equality saturates only when $m = 2$, the curves with singular multiplicities > 2 are of lower dimensions in the moduli space of genus g curves. A closer look at the type of the curve singularity shows that any double point other than nodal (A_1) singularity drops the complex dimension of the deformation space of curves by at least two. Therefore a generic genus g immersed curve dual to C develop $\delta = g(C) - g$ nodal singularities.

When one works in the C^∞ category and perturbs the almost complex structures of the algebraic surface M to a generic one, one expects the pseudo-holomorphic maps to develop nodal singularities. Thus both family Seiberg-Witten invariant of $C - 2\sum_{1\leq i\leq \delta} E_i$ and $Gr_{g(C)-\delta}(C)$ are objects enumerating δ-nodes curve dual to C.

The fundamental question we have to clarify and answer is,

Question: Do $\frac{1}{\delta!}\mathcal{AFSW}^*_{M_{\delta+1}\times T(M)\mapsto M_\delta\times T(M)}(1, C-\sum_{1\leq i\leq \delta} 2E_i)$ and the Ruan-Tian version of $GW_{g(C)-\delta}(C)$ "always" enumerate the "number of nodal curves" in a totally identical way?

Certainly there are many important cases that they do enumerate the same numbers, e.g. when C is a primitive cohomology class of a $K3$ or T^4, etc.

But the general answer of this question is "No". In the following we offer an explanation of the causes.

1. The Gromov-Witten invariant enumerates the "number of (pseudo)-holomorphic maps" instead of immersed curves (viewed as divisors in M). When the cohomology class C is primitive, i.e. it is not a multiple of any other element in $H^2(M, \mathbf{Z})$, each holomorphic map determines uniquely a nodal curve and vice versa. But when C becomes non-primitive, sometimes there can be multiple coverings of holomorphic maps such that the image (without counting multiplicity) is dual to $\frac{1}{m}C$, for some $m \in \mathbf{N}$. These multiple coverings of holomorphic maps contribute to $GW_{g(C)-\delta}(C)$ as well. But they do not correspond to immersed nodal curve dual to C and are mostly ignored by the

scheme of family invariant. In general, the object $GW_g(C)$ is $\mathbf{Q}$ valued, reflecting the orbifold structure of the space $\overline{\mathcal{M}_{g,n}}$. On the other hand, either $\frac{1}{\delta!}\mathcal{AFSW}^*_{M_{\delta+1}\times T(M)\mapsto M_\delta\times T(M)}(1, C-2\sum_{i\leq\delta}E_i)$ or $\frac{1}{\delta!}\mathcal{AFSW}^*_{M_{\delta+1}\times\{t_L\}\mapsto M_\delta\times\{t_L\}}(1, C-2\sum_{i\leq\delta}E_i)$ is always integral valued.

2. In the algebraic category, it is highly non-trivial to make the appropriated moduli space of curves defined as the zero locus of a transversal algebraic section. This applies to both family Seiberg-Witten invariant and Gromov-Witten invariant. When the moduli space is not transversal, one interprets the invariants as some types of virtual number counts. Without any transversality result, the correspondence between algebraic family Seiberg-Witten invariant and Gromov-Witten invariant is not transparent at all. If one decides to work instead in the symplectic (pseudo-holomorphic) category, it is usually easier to get the transversality result of the appropriated moduli spaces by choosing generic almost complex structures tamed by a simplectic structure on M. On the other hand, at this moment it is not clear how to define the "number of nodal singular curves" of a general symplectic four-manifold without selecting special classes C or almost complex structures J. In the situation when one can make sure the cut down moduli space consists of a finite number pseudo-holomorphic nodal singular curves, one has to identify the algebraic family Seiberg Witten invariant with its topological cousin "Family Seiberg-Witten invariant" and employ the technique of Taubes' "SW=Gr" to compare the solutions to family Seiberg-Witten equations and the smooth curve resolved from the nodal singular curves in M.

Unluckily the gluing machineries of Taubes in his seminal long papers [T1], [T2], falls out of the algebraic category. Thus one may desire a purely algebraic method to determine the family algebraic Seiberg-Witten invariants or relate them with the Gromov-Witten invariant.

3. In Göttsche's definition, the "number of nodal curves" is defined for L to be $5\delta-1-$very ample. Under this assumption, there is a well defined integer, gotten by counting the discrete number (with multiplicities) of nodal curves. On the other hand, when L fails to be $5\delta-1-$very ample, generally speaking we do not expect $\frac{1}{\delta}\mathcal{AFSW}^*_{M_{\delta+1}\times\{t_L\}\mapsto M_\delta\times\{t_L\}}(1, C-2\sum_{i\leq\delta}E_i)$ to calculate the number of nodal curves. In general, we have to subtract all the contributions from type II exceptional curves [Liu6] as well. The result is usually manifold dependent and involves some generalization of the technique used in this paper.

One exception is the case that $M=K3$ or T^4 when all the contributions from type II exceptional curves are known to vanish due to the fact $SW(2C)=0$ for any $C\in H^2(M,\mathbf{Z})$, $M=K3$ or $M=T^4$. The details about the contribution of the type II curve multiple-coverings will be developed in a separated article and we do not elaborate it here. In these cases, the numbers $\frac{1}{\delta!}\mathcal{AFSW}^*_{M_{\delta+1}\times T(M)\mapsto M_\delta\times T(M)}(1, C-2\sum_{i\leq\delta}E_i)$ are actually equal to "the number of nodal singular curves" dual to C, understood as a virtual intersection number. When C is primitive, this number coincides with the usual Gromov-Witten invariant ([BL1], [BL2]). When C is not primitive, the number of immersed nodal singular curves dual to C differ from the usual Gromov-

Witten invariant count as the former does not count the multiple covering maps of curves from the fractional multiples of C. This fact can be seen from the discrepancy between the Yau-Zaslow formula and Gathman's calculation [Gat] of $GW_g(C)$ of a 2-multiple of a primitive class in the $K3$ lattice.

In Gathman's calculation, he considers an algebraic K3 surface which is the double-covering of $\mathbf{P}^2$ ramified along a generic sextic curve. He considers the class C, $C^2 = 5$, to be twice of the pull-back of the hyperplane class from $\mathbf{P}^2$ and the answer he got for $GW_0(C)$ was $N_5 + \frac{1}{8}N_2$, where $\sum_{\delta\geq 0} N_\delta q^\delta = \prod_{i>0} \frac{1}{(1-q^i)^{24}}$ is the generating function of the Yau-Zaslow formula. In his calculation, he had used a degenerated complex structure to enumerate curves and the 176256 rational curves from his (i)., (ii)., and (iii)(b). can be thought to be the degenerations from nodal pseudo-holomorphic rational curves of generic $\mathbf{S}^2$ families of almost complex structures. On the other hand, the 324 rational curves from his (iii).(a). are honest double coverings of primitive rational curves and can not be degenerated from immersed pseudo-holomorphic nodal curves.

This example indicates clearly that Yau-Zaslow conjecture is not about the prediction of Gromov-Witten invariant at all. The prediction of Yau-Zaslow conjecture coincides with the Gromov-Witten calculation only for the primitive classes when the multiple coverings addressed in 1. do not occur. Therefore it is dangerous to mix up "the number of **nodal** rational curves" with "the number of holomorphic maps to rational curves" and identify them conceptually.

Moreover when the class C is not primitive, in Gathman's calculation the contributions from the multiple coverings does show up in the correction term and it is desirable to find out the relationship between them explicitly.

In the symplectic category if one can prevent the discrete number of pseudo-holomorphic maps with nodal curve images to converge to some multiple covering of curves in the most general content (this is not achieved at this moment), then a multiple covering formula (presumably determined by certain intersection numbers on $\mathcal{M}_{g,n}$) should allow us to relate the number of nodal singular curves dual to C with the numbers $GW_g(C)$ and the formula should be of combinatorial nature.

4. When the geometric genus p_g is greater than zero. The usual $GW_g(C)$ counts are mostly zero except for a finite number of classes (direct related to the so-called Seiberg-Witten basic classes for Kähler surfaces). Yet $\mathcal{AFSW}^*$ still picks up non-trivial contributions. This can be seen by the lower δ formula calculated by Vainsencher/Kleiman&Piene[][]. The reason behind the discrepancy is that $GW_g(C)$ (like Taubes' version of Gromov-Taubes invariant) are symplectic invariants. The algebraic surfaces are Seiberg-Witten simple type that classes with positive moduli space dimension $\frac{C^2 - c_1(\mathbf{K}_M)\cdot C}{2}$ have vanishing invariants. Algebraically it is reflected in the fact that there is a p_g difference between the dimension of the projective space $p_g + \frac{C^2 - C\cdot c_1(\mathbf{K}_M)}{2}$ (assuming $q(M) = 0$) and the expected Gromov-Taubes dimension of the curve $\frac{C^2 - C\cdot c_1(\mathbf{K}_M)}{2}$.

For a very ample line bundle L such that $\mathbf{K}_M \otimes L$ is ample, this gives a trivial rank p_g complex obstruction sub-bundle above $\mathcal{M}_L = |L|$, which causes

the usual Seiberg-Witten invariant of $c_1(L) = C$ and the sub-sequent family invariant of $C - \sum_{i\leq n} m_i E_i$ to vanish.

This is why the hyperwinding families of $K3$ or T^4 had been used to count the nodal curves in [Liu1]. The algebraic Seiberg-Witten invariants and the family algebraic Seiberg-Witten invariants are defined such that for such L, they remove the trivial rank p_g obstruction bundle from $|L|$ and shift the expected dimension of the moduli space up by p_g. This causes the algebraic family Seiberg-Witten invariants to be "enumeration invariants" but not the usual symplectic invariants under deformation. The way that we realize $\mathcal{ASW}$ or $\mathcal{AFSW}^*$, etc. to be invariants is through a different route: After calculating $\mathcal{ASW}$, $\mathcal{AFSW}$ or the modified invariants $\mathcal{AFSW}^*$, they can be expressed in terms of some datum which depend on only the homotopic type of the algebraic surface and C.

5. Another significant difference between the algebraic family Seiberg-Witten invariant of immersed curves and the Gromov-Witten invariant of maps is that (assuming $p_g = 0$ for simplicity) when the appropriated moduli spaces are transversal and the dimensions of the compactifying boundary components drop, the former object enumerates all the irreducible as well as reducible nodal singular curves dual to C while $GW_g(C)$ encodes only the irreducible nodal curves dual to C. The rough reason is that according to Taubes, $SW(2C - c_1(\mathbf{K}_M)) = Gr(C)$ enumerates the connected as well as disconnected smooth curves dual to C. Based on this philosophy that the enumeration of the family invariants of a given family should include connected as well as dis-connected smooth curves within a family, the modified family invariants of the universal families also enumerate reducible nodal curves where two curves intersect at normal crossing singularities (and the normal crossing singularities get resolved after the blowing ups). On the other hand, reducible nodal curves with multiple irreducible components can not be the pseudo-holomorphic image of irreducible Riemann surface. Thus, reducible nodal curves can only be viewed as the images of semi-stable maps while the source curve is parametrized by a point in the boundary point $\overline{\mathcal{M}_{g,n}} - \mathcal{M}_{g,n}$ for some pair of (g, n), and is of lower expected dimension.

This symptom is purely of combinatorial nature and should be cured by re-developing the generating series.

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